

AN ANALYTICAL CHARACTERIZATION OF EGUCHI–HANSON SPACE AND ITS HIGHER-DIMENSIONAL ANALOGS

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ABSTRACT. Let (M, g) be a complete 4-dimensional Ricci-flat ALE orbifold with group at infinity \mathbb{Z}_2 . We prove that if the L^2 kernel of its Lichnerowicz Laplacian has dimension at most 3, then (M, g) is either the Eguchi–Hanson space or the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$. A similar uniqueness result is proved for Calabi’s higher-dimensional analogs of the Eguchi–Hanson space among Ricci-flat Kähler ALE orbifolds with group at infinity \mathbb{Z}_m .

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1. INTRODUCTION

Ricci-flat asymptotically locally Euclidean (ALE) spaces lie at the intersection of differential geometry and mathematical physics. In geometry, they arise in the compactness theory and desingularization of Einstein metrics [3, 6, 7, 9, 18, 36, 40], as well as in the construction of manifolds with special holonomy [26, 27]. In physics, complete 4-dimensional Ricci-flat ALE spaces play a central role in Euclidean quantum gravity [19, 23] and form a class of gravitational instantons. For concreteness, we first lay out our definition of an ALE space, which allows for orbifold singularities in a compact set.

Definition 1.1. *An asymptotically locally Euclidean (ALE) space is a connected Riemannian orbifold (M^n, g) such that there is a finite subgroup of $\Gamma \subset SO(n)$ acting freely on \mathbb{S}^{n-1} , a compact set $K \subset M$, a radius $R > 0$, a diffeomorphism $\Phi : M \setminus K \rightarrow (\mathbb{R}^n \setminus B_R(0))/\Gamma$, and a decay rate $\tau > 0$ such that*

$$|\nabla^k(\Phi^*\bar{g} - g)|_g = \mathcal{O}(r^{-\tau-k})$$

as $r \rightarrow \infty$ for each $k \in \mathbb{N}_0$. Here, \bar{g} is the Euclidean metric on $(\mathbb{R}^n \setminus B_R(0))/\Gamma$, ∇ is the Levi-Civita connection with respect to g , and r is the Riemannian distance from a point with respect to g . We call τ the order of (M, g) , and Γ its group at infinity. If $\Gamma = \{1\}$, then (M, g) is called asymptotically Euclidean.

The prototypical example of a nontrivial (i.e. nonflat) 4-dimensional Ricci-flat ALE space is *Eguchi–Hanson space* [19]. This is a complete Ricci-flat ALE metric on the cotangent

bundle of \mathbb{S}^2 with group at infinity \mathbb{Z}_2 . The metric is hyperkähler and $U(2)$ -invariant with cohomogeneity one. Further details are provided in §2.2. We remark that Eguchi–Hanson space has been used by many authors to construct solutions to geometric PDEs via gluing; see for instance [9, 12, 18, 26].

Soon after Eguchi–Hanson space was discovered, Gibbons and Hawking [20] produced a large class of hyperkähler (thus Ricci-flat) gravitational instantons built from harmonic functions, including ALE spaces with groups at infinity \mathbb{Z}_k ($k \geq 1$) termed *multi-Eguchi–Hanson spaces*. The multi-Eguchi–Hanson spaces were also constructed by Hitchin using twistor methods [24]. Kronheimer’s work [32] then unified and extended the known constructions in the ALE case: he constructed and classified all simply-connected hyperkähler ALE 4-manifolds as hyperkähler quotients, exhibiting an ADE correspondence and identifying each space with the minimal resolution of \mathbb{C}^2/Γ for $\Gamma \subset SU(2)$.

Motivated by these developments, Bando, Kasue, and Nakajima [7] conjectured that every complete simply-connected Ricci-flat ALE 4-manifold (M, g) is hyperkähler and hence falls under Kronheimer’s classification. While this conjecture remains open in general, it has been shown to hold under additional assumptions. Nakajima [38] proved it in the affirmative assuming M is spin and $\Gamma \subset SU(2)$. Lock and Viaclovsky [35] proved that if M is diffeomorphic to a minimal resolution of \mathbb{C}^2/Γ or an iterated blowup thereof, where $\Gamma \subset U(2)$, then $\Gamma \subset SU(2)$ and g is hyperkähler. More recently, there has been progress on uniqueness of the Eguchi–Hanson space, which is a special case of the Bando–Kasue–Nakajima conjecture. Li [34] proved that if (M, g) is a 4-dimensional Ricci-flat ALE manifold with $\Gamma \subset SU(2)$ and admits a Hermitian non-Kähler structure, then (M, g) is Eguchi–Hanson space with the opposite orientation. Araneda and Lucietti [4] proved a similar uniqueness result assuming (M, g) is toric, Hermitian and non-Kähler, without any assumptions on $\Gamma \subset SO(4)$.

In this paper, we take a different angle. Rather than imposing algebraic structure or symmetry, we impose an analytical condition on the Lichnerowicz Laplacian and establish uniqueness in the class of Ricci-flat ALE spaces with certain groups at infinity. To state our results, we first define the Lichnerowicz Laplacian Δ_L on symmetric 2-tensors. On a Ricci-flat space this is given by

$$(1.1) \quad \Delta_L h = \Delta h + 2 \operatorname{Rm}(h),$$

where $(\Delta h)_{kl} = g^{ij} \nabla_{ij}^2 h_{kl}$ is the rough Laplacian and $\operatorname{Rm}(h)_{ik} = \operatorname{Rm}_{ijkl} h^{jl}$ is the action of the curvature tensor. The operator Δ_L is the linearized Ricci curvature in a suitable gauge, and its kernel elements are called *infinitesimal Einstein deformations* (IEDs) since they represent metric variations that preserve the Einstein condition to first order. In gluing problems for Einstein metrics, IEDs account for potential obstructions to resolving orbifold singularities by gluing in ALE manifolds. On ALE spaces, it is natural to confine our view to suitably decaying IEDs; for us this will be the space of L^2 -integrable IEDs $\ker_{L^2}(\Delta_L)$.

Our first result gives a classification in the 4-dimensional case when the group at infinity is \mathbb{Z}_2 and the space of L^2 -integrable IEDs is sufficiently small.

Theorem 1.2. *Let (M^4, g) be a complete 4-dimensional Ricci-flat ALE space with group at infinity \mathbb{Z}_2 . If $\dim \ker_{L^2}(\Delta_L) \leq 3$, then (M, g) is isometric to either the Eguchi–Hanson space or the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$.*

The flat $\mathbb{R}^4/\mathbb{Z}_2$ has $\dim \ker_{L^2}(\Delta_L) = 0$, whereas the threshold $\dim \ker_{L^2}(\Delta_L) = 3$ is achieved on Eguchi–Hanson space [9] (see also §2.2). By Theorem 1.2, there is nothing

in between. As a consequence, any counterexample to the Bando–Kasue–Nakajima conjecture with group at infinity \mathbb{Z}_2 must have $\dim \ker_{L^2}(\Delta_L) \geq 4$, and therefore carries a correspondingly subtler deformation theory.

Our second result is a similar theorem for Ricci-flat Kähler ALE spaces in higher dimensions. These are defined as follows.

Definition 1.3. *A Kähler orbifold (M^{2m}, g, J) of complex dimension m is called a Kähler ALE space if its underlying Riemannian orbifold (M, g) is an ALE space with group at infinity $\Gamma \subset U(m) \subset SO(2m)$.*

In particular, our definition does not require convergence of the complex structure J to the standard one on \mathbb{C}^m/Γ .

Among the earliest discovered Ricci-flat Kähler ALE spaces are what we call the *Calabi spaces*. The Calabi space of complex dimension $m \geq 2$ lives on the total space of the canonical line bundle $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$ of $\mathbb{C}\mathbb{P}^{m-1}$, and the group at infinity is $\mathbb{Z}_m \subset SU(m)$ generated by an m -th root of unity times the identity. It is equipped with a complete $U(m)$ -invariant cohomogeneity one Kähler metric. When $m = 2$, the Calabi space coincides with Eguchi–Hanson space and is therefore hyperkähler, but when $m \geq 3$ the Calabi space is not hyperkähler. Further details are provided in §2.3.

Morteza and Viaclovsky [36] computed that $\dim \ker_{L^2}(\Delta_L) = 1$ on all Calabi spaces of complex dimension $m \geq 3$. We will prove an analog of Theorem 1.2 for these spaces.

Theorem 1.4. *Let (M^{2m}, g, J) be a complete Ricci-flat Kähler ALE space with complex dimension $m \geq 3$ and group at infinity \mathbb{Z}_m . If*

$$\dim \ker_{L^2}(\Delta_L) \leq \begin{cases} 2m - 3 & \text{if } m \neq 4, \\ 2 & \text{if } m = 4, \end{cases}$$

then (M, g, J) is biholomorphically isometric to either a Calabi space or the flat Kähler orbifold $\mathbb{C}^m/\mathbb{Z}_m$, up to possible negation of the complex structure (i.e. $J \mapsto -J$).

This is a kind of gap theorem: no non-flat Ricci-flat Kähler ALE space asymptotic to $\mathbb{C}^m/\mathbb{Z}_m$ can have $1 < \dim \ker_{L^2}(\Delta_L) \leq 2m - 3$ (or $1 < \dim \ker_{L^2}(\Delta_L) \leq 2$ when $m = 4$). As we will explain shortly, the proof uses the classification of homogeneous metrics on lens spaces $\mathbb{S}^{2m-1}/\mathbb{Z}_m$ with large isometry groups. If the isometry group has dimension above some threshold, only Berger metrics can occur. The threshold is relatively higher when $m = 4$, so we impose a tighter bound on $\dim \ker_{L^2}(\Delta_L)$ in Theorem 1.4 to ensure the threshold is surpassed.

Outline of the paper. In §2, we lay out notations and conventions, then provide a brief summary of the Eguchi–Hanson and Calabi spaces.

In §3, we work on an arbitrary complete Ricci-flat ALE space with nontrivial group at infinity. We develop a symmetry principle for such spaces, showing that bounds on $\dim \ker_{L^2}(\Delta_L)$ imply the existence of global Killing fields tangential to a CMC foliation of the end. This generalizes a symmetry principle developed by the author for Ricci solitons with certain asymptotics [33], which was in turn based on Brendle’s seminal work on the Bryant soliton [10, 11].

In §4, we prove Theorem 1.2. Taking (M, g) as in the theorem, we use the results of §3 to find a 3-dimensional Lie group G acting isometrically on (M, g) . If G acts by cohomogeneity one, then we use well-known facts about homogeneous metrics on \mathbb{S}^3 and $\mathbb{R}\mathbb{P}^3$ to determine all possibilities for G . The structure theory of cohomogeneity one spaces then leads to a

diffeomorphism classification of M and a reduction of g to an ODE system. Aided by a theorem of Lock and Viaclovsky [35], this easily implies (M, g) is either Eguchi–Hanson or the flat $\mathbb{R}^4/\mathbb{Z}_2$. On the other hand, if G does not act by cohomogeneity one, then we show that it acts by cohomogeneity two. By analyzing the action further, we find that the end of (M, g) is (essentially) covered by a warped product with round \mathbb{S}^2 fibers; then by a variant of Birkhoff’s theorem in general relativity, the covering metric \tilde{g} is Euclidean Schwarzschild of mass m around an infinite ray. However, the ALE decay rate (Theorem 3.1) forces $m = 0$, so \tilde{g} (and hence g) is flat on an open set. By unique continuation, (M, g) must be the flat $\mathbb{R}^4/\mathbb{Z}_2$.

In §5, we prove Theorem 1.4. Taking (M, g) as in the theorem, we use the results of §3 to find a Lie group G of large dimension acting isometrically on (M, g) with cohomogeneity one. Using the classification of homogeneous metrics on certain lens spaces with large isometry groups, we show that the principal orbits of the action are quotient–Berger spheres. The structure theory of cohomogeneity one spaces then leads to a diffeomorphism classification of M and a reduction of g to an ODE system. If (M, g, J) is not the flat $\mathbb{C}^m/\mathbb{Z}_m$, we further use the Kähler condition and the form of g to determine J explicitly. This reduces everything to the Calabi ansatz, from which it follows that (M, g, J) is a Calabi space.

Acknowledgements. The author wishes to thank his advisor, William Minicozzi, for his continual encouragement and support. This work was carried out under the support of a Simons Dissertation Fellowship.

2. PRELIMINARIES

2.1. Notation and conventions. All Riemannian orbifolds are assumed to be smooth throughout. Given a Riemannian orbifold (M, g) , we will often abbreviate $g = \langle \cdot, \cdot \rangle$. Denote by ∇ the Levi-Civita connection, and $\Delta = \text{tr}_g \circ \nabla^2$ the Laplacian on tensor fields. Our convention for the Riemann curvature tensor is

$$\text{Rm}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z.$$

By the Myers–Steenrod theorem (see [5] for orbifolds), the isometry group $\mathcal{I}(M, g)$ of (M, g) is a Lie group acting smoothly on (M, g) . Its identity component is denoted by $\mathcal{I}^0(M, g)$. The Lie algebra of $\mathcal{I}(M, g)$, denoted by $\mathfrak{I}(M, g)$, is the space of Killing fields on (M, g) , i.e. vector fields X on M such that $\mathcal{L}_X g = 0$ where \mathcal{L} is the Lie derivative.

Given a Lie group G and a closed subgroup H of G , the centralizer and normalizer of H in G are denoted $C_G(H)$ and $N_G(H)$ respectively; both are closed subgroups of G . We let $C_G^0(H)$ and $N_G^0(H)$ be their identity components. The following facts will be used frequently. Their proofs are rudimentary and are therefore omitted.

Lemma 2.1. *If G is a Lie group and H is a discrete subgroup, then $N_G^0(H) = C_G^0(H)$.*

Lemma 2.2. *Let (\tilde{M}, \tilde{g}) be a simply-connected Riemannian manifold, and $\Gamma \subset \mathcal{I}(\tilde{M}, \tilde{g})$ a finite subgroup of isometries acting freely. Then the quotient manifold (M, g) has isometry group*

$$\mathcal{I}(M, g) = N_{\mathcal{I}(\tilde{M}, \tilde{g})}(\Gamma)/\Gamma.$$

2.2. Eguchi–Hanson space. Let $\{X_1, X_2, X_3\}$ be a left-invariant frame on $\mathbb{S}^3 \cong SU(2)$, and $\{\sigma_1, \sigma_2, \sigma_3\}$ its dual coframe. Let $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ be the Hopf fibration, chosen with fibers tangent to X_1 . For each $\varepsilon > 0$, define a metric $\tilde{g}_{\text{EH}, \varepsilon}$ on the manifold $(0, \infty) \times \mathbb{S}^3$ by

$$(2.1) \quad \tilde{g}_{\text{EH}, \varepsilon} = \frac{1}{2}(r^2 + \varepsilon)^{-\frac{1}{2}}(dr^2 + r^2\sigma_1 \otimes \sigma_1) + (r^2 + \varepsilon)^{\frac{1}{2}}(\sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3).$$

The metric $\tilde{g}_{\text{EH},\varepsilon}$ is invariant under left multiplication in $SU(2)$ and a right Hopf circle action. These are the only isometries connected to the identity; see Lemma 4.1. Thus, we have

$$(2.2) \quad \mathcal{I}^0((0, \infty) \times \mathbb{S}^3, \tilde{g}_{\text{EH},\varepsilon}) = SU(2) \times_{\mathbb{Z}_2} U(1) \cong U(2).$$

This acts with cohomogeneity one since it acts transitively on each level set of r .

In particular, $\tilde{g}_{\text{EH},\varepsilon}$ is invariant under the antipodal map on \mathbb{S}^3 (equivalently $q \mapsto -q$ on $SU(2)$), so it induces a metric $g_{\text{EH},\varepsilon}$ on $(0, \infty) \times \mathbb{R}\mathbb{P}^3$. Furthermore, as $r \rightarrow 0$, the Hopf fibers shrink and the level sets of r collapse to a round \mathbb{S}^2 . Thus, attaching an $\mathbb{S}^2 \cong \mathbb{C}\mathbb{P}^1$ at $r = 0$ results in a complete metric $g_{\text{EH},\varepsilon}$ on the smooth manifold $\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-2) \cong T^*\mathbb{S}^2$. We call $(\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-2), g_{\text{EH},\varepsilon})$ the ε -Eguchi–Hanson space. Changing ε only amounts to scaling the metric and applying a diffeomorphism, so we leave ε unspecified.

Since the antipodal map is central in the group (2.2), it follows from Lemma 2.2 that $\mathcal{I}^0((0, \infty) \times \mathbb{R}\mathbb{P}^3, g_{\text{EH},\varepsilon}) \cong U(2)/\mathbb{Z}_2$. These isometries also extend to the \mathbb{S}^2 bolt, so $\mathcal{I}^0(\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-2), g_{\text{EH},\varepsilon}) \cong U(2)/\mathbb{Z}_2$ as well. Thus, $U(2)$ acts isometrically and almost effectively on Eguchi–Hanson space with kernel \mathbb{Z}_2 . Moreover, as is readily checked from (2.1), Eguchi–Hanson space is Ricci-flat and ALE with group at infinity \mathbb{Z}_2 . It is in fact hyperkähler and self-dual.

It is well-known that $\ker_{L^2}(\Delta_L)$ has dimension 3 on Eguchi–Hanson space; see [9, 12] for an explicit basis. All elements of $\ker_{L^2}(\Delta_L)$ arise from Lie derivatives; one generator comes from scaling (i.e. changing ε), while the other two generators come from deformations of the complex structure. Thus, none of them integrate to nontrivial Einstein deformations.

2.3. Calabi space. Let $\pi : \mathbb{S}^{2m-1} \rightarrow \mathbb{C}\mathbb{P}^{m-1}$ be the Hopf fibration, where $m \geq 2$. Let g_{FS} be the Fubini–Study metric on $\mathbb{C}\mathbb{P}^{m-1}$ with holomorphic sectional curvature 4, so that the round metric on \mathbb{S}^{2m-1} of sectional curvature 1 splits as

$$g_{\mathbb{S}^{2m-1}} = \sigma \otimes \sigma + \pi^* g_{\text{FS}},$$

where σ is a 1-form dual to the Hopf vector field ξ , satisfying $d\sigma = 2\pi^*\omega_{\text{FS}}$. For $a, b > 0$, define the *Berger metric* on \mathbb{S}^{2m-1} by

$$\tilde{g}_{a,b} = a^2 \sigma \otimes \sigma + b^2 \pi^* g_{\text{FS}}.$$

It is standard that $\mathcal{I}^0(\mathbb{S}^{2m-1}, \tilde{g}_{a,b}) \cong U(m)$ if $a \neq b$, and $\mathcal{I}^0(\mathbb{S}^{2m-1}, \tilde{g}_{a,b}) \cong SO(2m)$ if $a = b$.

The cyclic subgroup $\mathbb{Z}_m \subset U(1) \subset U(m)$ acts freely on \mathbb{S}^{2m-1} by rotation along the Hopf fibers. We write

$$\mathbb{S}^{2m-1}/\mathbb{Z}_m$$

to denote the resulting lens space. The \mathbb{Z}_m -action is isometric on $(\mathbb{S}^{2m-1}, \tilde{g}_{a,b})$ for any $a, b > 0$, and is central in its isometry group, so $\tilde{g}_{a,b}$ descends to a *quotient-Berger metric* $g_{a,b}$ on $\mathbb{S}^{2m-1}/\mathbb{Z}_m$. By Lemma 2.2, we have $\mathcal{I}^0(\mathbb{S}^{2m-1}/\mathbb{Z}_m, g_{a,b}) \cong U(m)/\mathbb{Z}_m$ whenever $a \neq b$.

Consider the complex line bundle $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$ over $\mathbb{C}\mathbb{P}^{m-1}$, which is isomorphic to the canonical line bundle of $\mathbb{C}\mathbb{P}^{m-1}$. We now briefly review Calabi’s construction [13] of Ricci-flat Kähler ALE metrics on $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$ using what is now called the *Calabi ansatz*; a detailed account may be found elsewhere (e.g. [42, §8]). We first note that $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$ has sphere bundles diffeomorphic to $\mathbb{S}^{2m-1}/\mathbb{Z}_m$. Thus, choosing a radial coordinate r along the fibers, we obtain a diffeomorphism

$$(2.3) \quad \mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m) \setminus \mathbb{C}\mathbb{P}^{m-1} \cong (0, \infty) \times \mathbb{S}^{2m-1}/\mathbb{Z}_m.$$

After identifying the complement of the zero section in $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$ via (2.3), one considers metrics of the form

$$(2.4) \quad g = u(r)\pi^*g_{\text{FS}} + v(r)(dr^2 + r^2\sigma \otimes \sigma) = v(r)dr^2 + g_{r\sqrt{v(r)}, \sqrt{u(r)}},$$

where u and v are smooth positive functions. Equivalently, with the canonical complex structure under which $J(\partial_r) = \frac{1}{r}\xi$, one may consider Kähler forms on $(0, \infty) \times \mathbb{S}^{2m-1}/\mathbb{Z}_m$ of the form

$$(2.5) \quad \omega = u(r)\pi^*\omega_{\text{FS}} + v(r)r dr \wedge \sigma,$$

where ω_{FS} is the Fubini–Study Kähler form on $\mathbb{C}\mathbb{P}^{m-1}$. The Kähler condition $d\omega = 0$ and the Ricci-flatness condition yield, respectively,

$$(2.6) \quad u' = 2rv, \quad u^{m-1}v = 2.$$

Solving the system (2.6), we get the general solution

$$(2.7) \quad u(r) = (2m(r^2 + \varepsilon))^{\frac{1}{m}}, \quad v(r) = 2(2m(r^2 + \varepsilon))^{-\frac{m-1}{m}}$$

where $\varepsilon \in \mathbb{R}$. Let $g_{\text{Cal}, \varepsilon}$ be the metric (2.4) with u and v given by (2.7). If $\varepsilon > 0$, then $g_{\text{Cal}, \varepsilon}$ closes up smoothly at $r = 0$ to give a complete metric on $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$, also denoted by $g_{\text{Cal}, \varepsilon}$. We call $(\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m), g_{\text{Cal}, \varepsilon})$ the ε -Calabi space. Changing ε only amounts to scaling the metric and applying a diffeomorphism, so we leave ε unspecified.

It follows from the above discussion about isometry groups of Berger metrics that $\mathcal{I}^0(\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m), g_{\text{Cal}, \varepsilon}) \cong U(m)/\mathbb{Z}_m$. Thus, $U(m)$ acts isometrically and almost effectively on Calabi space with kernel \mathbb{Z}_m . The action preserves and acts transitively on each level set of r , so it has cohomogeneity one. Moreover, as is readily checked from (2.4) and (2.7), Calabi space is Ricci-flat and Kähler ALE with group at infinity \mathbb{Z}_m . When $m = 2$, Calabi space coincides with Eguchi–Hanson space and is therefore hyperkähler. On the other hand, the Calabi spaces with $m \geq 3$ are not hyperkähler.

Finally, $\ker_{L^2}(\Delta_L)$ is one-dimensional on each Calabi space with $m \geq 3$. Indeed, this follows from [36, Theorem 2.2], noting that if $h \in \ker_{L^2}(\Delta_L)$ on a complete Ricci-flat manifold, then $\text{tr}_g h \in \ker_{L^2}(\Delta) = \{0\}$. The generating element of $\ker_{L^2}(\Delta_L)$ comes from scaling (i.e. changing ε), so it does not integrate to a nontrivial Einstein deformation.

3. GLOBAL SYMMETRY PRINCIPLES

3.1. Geometry of ALE spaces. First, we have the following result about decay rates of Ricci-flat ALE spaces, due to Kröncke and Szábo (see also [7, 14]). The original proof is given for Ricci-flat ALE manifolds, but it applies to Ricci-flat ALE spaces after standard orbifold adaptations of the weighted elliptic and gauge-fixing theory.

Theorem 3.1 ([31]). *Let (M^n, g) be an n -dimensional Ricci-flat ALE space with group at infinity $\Gamma \neq \{1\}$. Then (M, g) is of order n .*

We will also use that every ALE space with nontrivial group at infinity admits an end foliation by constant mean curvature (CMC) hypersurfaces. This follows from work of Chodosh, Eichmair, and Volkman.

Theorem 3.2 ([15]). *Let (M^n, g) be an ALE space with group at infinity $\Gamma \neq \{1\}$. Then there is a canonical CMC foliation of the end of (M, g) by smooth hypersurfaces diffeomorphic to \mathbb{S}^{n-1}/Γ .*

Each leaf of the foliation bounds an isoperimetric region of some volume $V > 0$. This is the unique isoperimetric region of volume V in (M, g) .

Let H be the function sending a point x on the end of M to the mean curvature of the leaf containing x . Then H is a smooth function and ∇H is nonvanishing.

Corollary 3.3. *Let (M, g) be an ALE space with group at infinity $\Gamma \neq \{1\}$. If W is a Killing field on (M, g) , then W is tangential to the leaves of the canonical CMC foliation, i.e. $\langle W, \nabla H \rangle = 0$. Moreover, $[W, \nabla H] = 0$.*

Proof. Let Σ be a leaf of the CMC foliation which bounds an isoperimetric region of some volume V . Let Φ_t be the time t flow of W . Since W is a Killing field, $\Phi_t(\Sigma)$ also bounds an isoperimetric region of volume V . By the uniqueness part of Theorem 3.2, $\Phi_t(\Sigma) = \Sigma$. This implies that W is tangential to Σ , i.e. $\langle W, \nabla H \rangle = 0$.

Next, we recall a general identity for vector fields X and Y :

$$\mathcal{L}_X(Y^\flat) = (\mathcal{L}_X Y)^\flat + (\mathcal{L}_X g)(Y, \cdot).$$

Applying this with $X = W$ and $Y = \nabla H$ gives

$$[W, \nabla H] = \mathcal{L}_W(\nabla H) = (\mathcal{L}_W dH)^\sharp = (d\mathcal{L}_W H)^\sharp = (d\langle W, \nabla H \rangle)^\sharp = 0,$$

which completes the proof. \square

3.2. Finding global Killing fields. For the rest of this section, we let (M^n, g) be a complete Ricci-flat ALE space of dimension $n \geq 4$ with group at infinity $\Gamma \neq \{1\}$. We write $\mathcal{S} := \mathbb{S}^{n-1}/\Gamma$, and \bar{g}_∞ for the round metric on \mathcal{S} with constant sectional curvature 1. Also fix an ALE diffeomorphism Φ as in Definition 1.1. By Theorem 3.1, we may assume that for each $k \geq 0$,

$$(3.1) \quad |\nabla^k(\Phi^*\bar{g} - g)|_g = \mathcal{O}(r^{-n-k}).$$

Given a vector field \bar{U} on \mathcal{S} , we can radially extend it to a vector field on $(\mathbb{R}^n \setminus B_R(0))/\Gamma$ also denoted by \bar{U} . We will say that a vector field U on M extends \bar{U} if $\Phi_*U = \bar{U}$ on $(\mathbb{R}^n \setminus B_R(0))/\Gamma$.

Proposition 3.4. *Let \bar{U} be a Killing field on $(\mathcal{S}, \bar{g}_\infty)$, and let U be a smooth vector field on M which extends \bar{U} . Then there is a smooth vector field W on M such that*

$$(3.2) \quad \Delta W = 0,$$

$$(3.3) \quad |\nabla^k(W - U)| \leq \mathcal{O}(r^{-n+2+\varepsilon-k}) \text{ for each } k \geq 0,$$

$$(3.4) \quad |\nabla^k W| \leq \mathcal{O}(r^{1-k}) \text{ for each } k \geq 0,$$

$$(3.5) \quad |\mathcal{L}_W g| \leq \mathcal{O}(r^{-n+1+\varepsilon}).$$

Proof. Let Φ and \bar{g} be as in Definition 1.1, and let $h = \Phi^*\bar{g} - g$. The radial extension of \bar{U} (still denoted \bar{U}) satisfies $|\bar{\nabla}^k \bar{U}|_{\bar{g}} = \mathcal{O}(r^{1-k})$ for each $k \geq 0$, where $\bar{\nabla}$ is the Euclidean derivative. Using (3.1), one can deduce

$$(3.6) \quad |\nabla^k U| \leq \mathcal{O}(r^{1-k}).$$

Using (3.1) and (3.6), we also have

$$(3.7) \quad |\nabla^k \mathcal{L}_U g| = |\nabla^k \mathcal{L}_U h| = |\nabla^k(\nabla U * h + U * \nabla h)| \leq \mathcal{O}(r^{-n-k}) \text{ for each } k \geq 0.$$

Next, recall the general relation

$$\operatorname{div}(\mathcal{L}_U g) - \frac{1}{2}\nabla(\operatorname{tr} \mathcal{L}_U g) = \Delta U + \operatorname{Ric}(U).$$

By (3.7) and Ricci-flatness, this implies

$$(3.8) \quad |\nabla^k \Delta U| \leq C|\nabla^{k+1} \mathcal{L}_U g| \leq \mathcal{O}(r^{-n-1-k}) \text{ for all } k \geq 0.$$

For $k \geq 0$, $\alpha \in (0, 1)$, and $\beta \in \mathbb{R}$, let $C_\beta^{k, \alpha}$ be the weighted Hölder space of vector fields X on M such that

$$\|X\|_{C_\beta^{k, \alpha}} := \sum_{j=0}^k \left(\sup_{x \in M} r(x)^{j-\beta} |\nabla^j X(x)| \right) + \sup_{x \in M} r(x)^{k+\alpha-\beta} |\nabla^k X|_{C^\alpha(B_{r(x)/2}(x))}$$

is finite. By standard elliptic theory on ALE spaces (see e.g. [31, §4.1]), we have that for all $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$, and all $\beta \in \mathbb{R}$ outside a discrete exceptional set $\mathcal{D} \subset \mathbb{R}$,

$$(3.9) \quad \Delta : C_{\beta+2}^{k+2, \alpha} \rightarrow C_\beta^{k, \alpha}$$

is Fredholm with image

$$\left\{ Y \in C_\beta^{k, \alpha} : \int_M \langle Y, X \rangle = 0 \text{ for all } X \in \mathcal{H}_{-\beta-n} \right\},$$

where $\mathcal{H}_{-\beta-n}$ is the space of harmonic vector fields in $C_{-\beta-n}^{\ell, \alpha}$ for arbitrary ℓ . We let $\beta = -n + \varepsilon$, where $\varepsilon > 0$ is chosen so that $\beta \notin \mathcal{D}$. Then any $X \in \mathcal{H}_{-\beta-n}$ satisfies $|X| = \mathcal{O}(r^{-\varepsilon})$ and is therefore in L^p for some $p \in (1, \infty)$. By [30, Proposition 4.3], this promotes to $|X| = \mathcal{O}(r^{-n+1})$. Thus X is an L^2 harmonic vector field, which must be zero by the Bochner formula and an integration-by-parts argument. Hence, (3.9) is surjective for $\beta = -n + \varepsilon$.

By (3.8), we have $-\Delta U \in C_{-n+\varepsilon}^{k, \alpha}$ for all $k \geq 0$ and $\alpha \in (0, 1)$. Hence there is a vector field $\hat{U} \in C_{-n+2+\varepsilon}^{2, \alpha}$ such that

$$\Delta \hat{U} = -\Delta U.$$

Standard weighted elliptic estimates imply that $\hat{U} \in C_{-n+2+\varepsilon}^{k+2, \alpha}$ for all $k \geq 0$. So $W := U + \hat{U}$ satisfies (3.2) and (3.3). Combining with (3.6) and (3.7), we obtain (3.4) and (3.5) respectively. \square

Recall that the Lichnerowicz Laplacian Δ_L is defined by (1.1) on a Ricci-flat space. The following identity is well-known: see for instance [10, Theorem 4.1] (taking $X = 0$ there).

Lemma 3.5. *On a Ricci-flat space (M, g) , if W is a vector field satisfying $\Delta W = 0$, then $\Delta_L(\mathcal{L}_W g) = 0$.*

Next, we show that a dimension bound on $\ker_{L^2}(\Delta_L)$ implies the existence of Killing fields on (M, g) . Below, we denote by $\mathcal{I}(M, g)$ the isometry group of (M, g) , $\mathcal{I}^0(M, g)$ its identity component, and $\mathfrak{J}(M, g)$ its Lie algebra (i.e. the Lie algebra of Killing fields). Define $\bar{d} = \dim \mathcal{I}(\mathcal{S}, \bar{g}_\infty)$ to be the dimension of the isometry group of the asymptotic link.

Proposition 3.6. *Suppose $\dim \ker_{L^2}(\Delta_L) \leq d$ for some nonnegative integer d . Then there exists a $(\bar{d} - d)$ -dimensional vector space \mathcal{V} of Killing fields on (M, g) such that for each $W \in \mathcal{V}$,*

- (a) $[W, \nabla H] = 0$.
- (b) $\langle W, \nabla H \rangle = 0$, i.e. W is tangent to the CMC leaves.
- (c) W is a Killing field for (M, g) and for each sufficiently far CMC leaf.
- (d) $|\nabla^k W| \leq \mathcal{O}(r^{1-k})$ for each $k \geq 0$.
- (e) There exists a unique Killing field \bar{U} on $(\mathcal{S}, \bar{g}_\infty)$ such that if U is a smooth vector field on M which extends \bar{U} , then $|\nabla^k(W - U)| \leq \mathcal{O}(r^{-n+2+\varepsilon-k})$ for each $k \geq 0$. The map $\mathcal{V} \rightarrow \mathfrak{J}(\mathcal{S}, \bar{g}_\infty)$, $W \mapsto \bar{U}$ is linear and injective.

Proof. Let $\{\bar{U}_i\}_{i=1}^{\bar{d}}$ be a basis for the space of Killing fields $\mathfrak{J}(\mathcal{S}, \bar{g}_\infty)$, and let $\{U_i\}_{i=1}^{\bar{d}}$ be smooth vector fields on M which extend \bar{U}_i . Proposition 3.4 gives vector fields $\{W_i\}_{i=1}^{\bar{d}}$ such that

$$(3.10) \quad \Delta W_i = 0,$$

$$(3.11) \quad |\nabla^k(W_i - U_i)| \leq \mathcal{O}(r^{-n+2+\varepsilon-k}) \text{ for each } k \geq 0,$$

$$(3.12) \quad |\nabla^k W_i| \leq \mathcal{O}(r^{1-k}) \text{ for each } k \geq 0,$$

$$(3.13) \quad |\mathcal{L}_{W_i} g| \leq \mathcal{O}(r^{-n+1+\varepsilon}).$$

By (3.10), (3.13) and Lemma 3.5, we have $\mathcal{L}_{W_i} g \in \ker_{L^2}(\Delta_L)$. By (3.11) with $k = 0$, the vector fields W_i are clearly linearly independent. It is also clear that for each i , U_i is the unique vector field which is both an extension of a Killing field on $(\mathcal{S}, \bar{g}_\infty)$ and satisfies (3.11) (for this fixed W_i).

Let $\mathcal{V}' = \text{span}\{W_i\}_{i=1}^{\bar{d}}$, so that $\dim \mathcal{V}' = \bar{d}$ by linear independence. Define a map $\mathcal{R} : \mathcal{V}' \rightarrow \mathfrak{J}(\mathcal{S}, \bar{g}_\infty)$ by sending W_i to \bar{U}_i and extending linearly. This map is clearly injective. Also define a map $\mathcal{T} : \mathcal{V}' \rightarrow \ker_{L^2}(\Delta_L)$ by $W \mapsto \mathcal{L}_W g$.

The assumption $\dim \ker_{L^2}(\Delta_L) \leq d$ implies $\dim \ker \mathcal{T} \geq \bar{d} - d$. Let $\mathcal{V} = \ker \mathcal{T}$. Then every $W \in \mathcal{V}$ is a Killing field for (M, g) . Combined with Corollary 3.3, this proves (a), (b), and (c). Part (d) follows from (3.12). Part (e) follows from the uniqueness part of (3.11), and by restricting the map $\mathcal{R} : \mathcal{V}' \rightarrow \mathfrak{J}(\mathcal{S}, \bar{g}_\infty)$ to \mathcal{V} . \square

For all large $\rho > 0$, let $\Sigma_\rho \subset M$ be the CMC leaf with mean curvature $\frac{n-1}{\rho}$. We will identify Σ_ρ (with its induced metric) with a Riemannian manifold (\mathcal{S}, g_ρ) . Let \mathcal{V}_ρ be the restriction of Killing fields in \mathcal{V} to the level set Σ_ρ . We have $\mathcal{V}_\rho \subset \mathfrak{J}(\mathcal{S}, g_\rho)$ by Proposition 3.6. Furthermore, let \mathfrak{g} be the Lie subalgebra of $\mathfrak{J}(M, g)$ generated by \mathcal{V} , and let G be the unique connected Lie subgroup of $\mathcal{I}^0(M, g)$ generated by \mathfrak{g} . We define $\mathfrak{g}_\rho \subseteq \mathfrak{J}(\mathcal{S}, g_\rho)$ and $G_\rho \subseteq \mathcal{I}^0(\mathcal{S}, g_\rho)$ analogously.

By Corollary 3.3, we have:

Corollary 3.7. *For any $X \in \mathfrak{g}$, we have $[X, \nabla H] = 0$ and $\langle X, \nabla H \rangle = 0$.*

With the help of Corollary 3.7, the following assertions are proved in exactly the same manner as [33, Lemma 3.19]. Thus we omit the details.

Lemma 3.8. (a) *The restriction map $\mathcal{V} \rightarrow \mathcal{V}_\rho$ is a linear isomorphism.*

(b) *The restriction map $\mathfrak{g} \rightarrow \mathfrak{g}_\rho$ is a Lie algebra isomorphism.*

(c) *If G_ρ is closed in $\mathcal{I}(\mathcal{S}, g_\rho)$, then G is a compact Lie group acting smoothly and isometrically on (M, g) , and the restriction map $G \rightarrow G_\rho$ is a Lie group isomorphism.*

In the case where $(\mathcal{S}, \bar{g}_\infty)$ is homogeneous, i.e. its isometry group acts transitively, the space of Killing fields $\mathfrak{J}(\mathcal{S}, \bar{g}_\infty)$ spans the tangent space to \mathcal{S} at each point. We define the quantity $d_1(\mathcal{S}, \bar{g}_\infty) \geq 0$ to be the largest number such that

- Any Lie subalgebra of $\mathfrak{J}(\mathcal{S}, \bar{g}_\infty)$ with dimension $\geq \bar{d} - d$ spans the tangent space to \mathcal{S} at every point.

Lemma 3.9. *If $(\mathcal{S}, \bar{g}_\infty)$ is homogeneous and $\dim \ker_{L^2}(\Delta_L) \leq d_1(\mathcal{S}, \bar{g}_\infty)$, then \mathfrak{g}_ρ spans the tangent space to Σ_ρ at every point for all sufficiently large ρ .*

Proof. Proposition 3.6 gives a $(\bar{d} - d_1(\mathcal{S}, \bar{g}_\infty))$ -dimensional space \mathcal{V} of Killing fields on (M, g) . Take a basis $\{W_i\}_{i=1, \dots, \bar{d}-d_1(\mathcal{S}, \bar{g}_\infty)}$ for \mathcal{V} , let \bar{U}_i be the corresponding vector fields from

Proposition 3.6(e), and let U_i be arbitrary extensions thereof. Since $W_i \mapsto \bar{U}_i$ is linear and injective, it follows that U_i are linearly independent.

Write $X_i = W_i - U_i$. From Propositions 3.4 and 3.6, we have

$$|\nabla^\ell W_i| \leq \mathcal{O}(r^{1-\ell}), \quad |\nabla^\ell X_i| \leq \mathcal{O}(r^{-n+2+\varepsilon-\ell}), \quad |\nabla^\ell U_i| \leq \mathcal{O}(r^{1-\ell})$$

for each $\ell \geq 0$. The rest of the argument is then identical to [33, Lemma 3.21], so we omit it here. \square

The next proposition elucidates the global structure of (M, g) if it admits an isometric action of cohomogeneity one. It is an orbifold version of [33, Proposition 3.22], which follows from the work of Mostert [37] in the setting of smooth manifolds and González Álvaro for smooth orbifolds [21].

Proposition 3.10. *Suppose there is a compact, connected Lie group G acting smoothly, almost effectively, and isometrically with cohomogeneity one on (M, g) . Denote the projection map by $\pi : M \rightarrow M/G$. Then:*

- (a) *The orbit space M/G is homeomorphic to $[0, 1)$.*
- (b) *There is a closed subgroup H of G such that the principal orbits, i.e. $\pi^{-1}(s)$ for any $s > 0$, are diffeomorphic to $\mathcal{S} \cong G/H$. Thus, each point in a principal orbit has isotropy subgroup H .*
- (c) *The singular orbit $X = \pi^{-1}(0)$ is diffeomorphic to G/K , where K is a closed subgroup of G such that $H \subset K$ and $K/H \cong \mathcal{S}'$ for some spherical space form \mathcal{S}' of dimension $\ell \geq 0$. Moreover, M is a smooth manifold if and only if \mathcal{S}' is a sphere.*
- (d) *M is diffeomorphic to the orbifold bundle $E = G \times_K \text{Cone}(\mathcal{S}')$ over $G/K \cong X$, where K acts via $k \cdot (g, v) = (gk^{-1}, kv)$, and kv is the radial extension of the action of K on $K/H \cong \mathcal{S}'$.*
- (e) *Every sphere bundle of E is diffeomorphic to \mathcal{S} . Thus $M \setminus X$ is diffeomorphic to $(0, \infty) \times \mathcal{S}$, and under this identification the metric g becomes*

$$g = ds^2 + g(s),$$

where $\{g(s)\}_{s>0}$ is a smooth family of G -invariant metrics on $G/H \cong \mathcal{S}$.

4. A CHARACTERIZATION OF EGUCHI–HANSON SPACE

In this section, we prove Theorem 1.2. Before proceeding, we require some preparations regarding metrics on \mathbb{S}^3 and \mathbb{RP}^3 with large isometry groups.

4.1. Metrics on \mathbb{S}^3 and \mathbb{RP}^3 with large isometry groups. Below, we will identify \mathbb{S}^3 as the multiplicative group $SU(2)$ of unit quaternions

$$\mathbb{S}^3 \cong SU(2) = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}.$$

Any left-invariant metric g on $SU(2)$ is of the form

$$(4.1) \quad \tilde{g} = \lambda^2 \sigma_1 \otimes \sigma_1 + \mu^2 \sigma_2 \otimes \sigma_2 + \nu^2 \sigma_3 \otimes \sigma_3,$$

where $\lambda, \mu, \nu > 0$ are constants, and $\{\sigma_1, \sigma_2, \sigma_3\}$ is dual to a left-invariant frame $\{X_1, X_2, X_3\}$. We may assume $X_1 = \mathbf{i}$, $X_2 = \mathbf{j}$, $X_3 = \mathbf{k}$ in $T_1 SU(2) = \mathfrak{su}(2)$. If $\lambda = \mu = \nu$, then \tilde{g} is a round metric. If exactly two coefficients are equal, then we call g a *Berger metric*. If all three coefficients are different, then we call \tilde{g} a *generic left-invariant metric*. The corresponding isometry groups are well-known (see for instance [22]) and are stated in the next lemma.

Lemma 4.1. *Let \tilde{g} be a homogeneous metric on \mathbb{S}^3 . Then \tilde{g} is a left-invariant metric on $SU(2)$, hence is of the form (4.1). The isometry group $\mathcal{I}(SU(2), \tilde{g})$ is given as follows:*

(a) If \tilde{g} is a round metric ($\lambda = \mu = \nu$), then $\mathcal{I}(SU(2), \tilde{g})$ is generated by left multiplications in $SU(2)$, right multiplications in $SU(2)$, and inversion $q \mapsto q^{-1}$ in $SU(2)$. Thus,

$$\mathcal{I}(SU(2), \tilde{g}) = (SU(2)_L \times_{\mathbb{Z}_2} SU(2)_R) \rtimes \mathbb{Z}_2 \cong SO(4) \rtimes \mathbb{Z}_2 \cong O(4).$$

(b) If \tilde{g} is a Berger metric (without loss of generality we assume $\lambda \neq \mu = \nu$), then $\mathcal{I}(SU(2), \tilde{g})$ is generated by left multiplications in $SU(2)$, right multiplications by $e^{i\theta}$, and the map $q \mapsto \mathbf{j}^{-1}q\mathbf{j}$. Thus,

$$\mathcal{I}(SU(2), \tilde{g}) = (SU(2)_L \times_{\mathbb{Z}_2} U(1)_R) \rtimes \mathbb{Z}_2 \cong U(2) \rtimes \mathbb{Z}_2.$$

(c) If λ, μ, ν are distinct, then $\mathcal{I}(SU(2), \tilde{g})$ is generated by left multiplications in $SU(2)$, the map $q \mapsto \mathbf{i}^{-1}q\mathbf{i}$, and the map $q \mapsto \mathbf{j}^{-1}q\mathbf{j}$. Thus,

$$\mathcal{I}(SU(2), \tilde{g}) = SU(2)_L \rtimes (\mathbb{Z}_2)^2.$$

Lemma 4.1 leads to a classification of homogeneous metrics on \mathbb{RP}^3 .

Corollary 4.2. *If g is a homogeneous metric on \mathbb{RP}^3 , then g is isometric to the quotient of a homogeneous metric \tilde{g} on $SU(2)$ by the map $q \mapsto -q$. We have $\mathcal{I}(\mathbb{RP}^3, g) = \mathcal{I}(SU(2), \tilde{g})/\mathbb{Z}_2$ and $\mathcal{I}^0(\mathbb{RP}^3, g) = \mathcal{I}^0(SU(2), \tilde{g})/\mathbb{Z}_2$.*

Proof. Let g be as in the corollary. Then the universal cover $(SU(2), \tilde{g})$ is homogeneous, and there is an embedding $\iota : \mathbb{Z}_2 \hookrightarrow \mathcal{I}(SU(2), \tilde{g})$ such that $\iota(\mathbb{Z}_2)$ acts freely and isometrically on $(SU(2), \tilde{g})$ with quotient isometric to (\mathbb{RP}^3, g) . Note that in each case of Lemma 4.1, the center of $\mathcal{I}^0(SU(2), \tilde{g})$ contains a unique copy of \mathbb{Z}_2 generated by the antipodal map $q \mapsto -q$. Therefore, the assertions follow as long as $\iota(\mathbb{Z}_2)$ is central in $\mathcal{I}^0(SU(2), \tilde{g})$. We do this for each case (a)–(c) in Lemma 4.1.

(a) If \tilde{g} is round, then so is g . Since diffeomorphic spherical space forms are isometric [17], and a round \mathbb{RP}^3 can be obtained by quotienting $(SU(2), \tilde{g})$ by the map $q \mapsto -q$, it follows that (\mathbb{RP}^3, g) is isometric to the ‘usual’ round \mathbb{RP}^3 .

(b) If \tilde{g} is a Berger metric, then $\mathcal{I}(SU(2), \tilde{g}) \cong U(2) \rtimes \mathbb{Z}_2$. By Lemma 2.1 and Lemma 2.2, we have

$$(4.2) \quad \dim C_{\mathcal{I}(SU(2), \tilde{g})}^0(\iota(\mathbb{Z}_2)) = \dim N_{\mathcal{I}(SU(2), \tilde{g})}^0(\iota(\mathbb{Z}_2)) = \dim \mathcal{I}^0(\mathbb{RP}^3, g) \geq 3,$$

where the last inequality holds because g is homogeneous. We have $C_{\mathcal{I}(SU(2), \tilde{g})}^0(\iota(\mathbb{Z}_2)) \subset \mathcal{I}^0(SU(2), \tilde{g}) \cong U(2)$, so (4.2) implies that $C_{\mathcal{I}(SU(2), \tilde{g})}^0(\iota(\mathbb{Z}_2))$ is either $SU(2)$ or $U(2)$. Either way, it follows that $\iota(\mathbb{Z}_2) \subset C_{\mathcal{I}(SU(2), \tilde{g})}(SU(2))$. This centralizer is computed to be isomorphic to $Z(\mathcal{I}^0(SU(2), \tilde{g})) \rtimes \mathbb{Z}_2 \cong \text{Pin}(2)$. Since the unique order 2 subgroup of $\text{Pin}(2)$ is contained in its identity component, it follows that $\iota(\mathbb{Z}_2) \subset Z(\mathcal{I}^0(SU(2), \tilde{g}))$, as required.

(c) If \tilde{g} is a generic left-invariant metric, then $\mathcal{I}(SU(2), \tilde{g}) \cong SU(2) \rtimes (\mathbb{Z}_2)^2$. The inequality (4.2) still holds, but now it implies $C_{\mathcal{I}(SU(2), \tilde{g})}^0(\iota(\mathbb{Z}_2)) = SU(2)$. Thus,

$$\iota(\mathbb{Z}_2) \subset C_{\mathcal{I}(SU(2), \tilde{g})}(SU(2)) = Z(SU(2)) = Z(\mathcal{I}^0(SU(2), \tilde{g})),$$

where the first equality is easily checked using Case (c) of Lemma 4.1. \square

The next lemma follows from Lemma 4.1 together with standard facts about subgroups of $SO(4)$ and $U(2)$.

Lemma 4.3. *Let \tilde{g} be a homogeneous metric on \mathbb{S}^3 . For each case below, we list all Lie subalgebras \mathfrak{g} of $\mathcal{J}(\mathbb{S}^3, \tilde{g})$ with $\dim \mathfrak{g} \geq 3$, up to conjugacy in $\mathcal{I}^0(\mathbb{S}^3, \tilde{g})$. For each \mathfrak{g} , we identify the unique connected Lie subgroup G of $\mathcal{I}^0(\mathbb{S}^3, \tilde{g})$ with Lie algebra \mathfrak{g} . This gives a complete list of connected Lie subgroups $G \subseteq \mathcal{I}^0(\mathbb{S}^3, \tilde{g})$ with $\dim G \geq 3$, up to conjugacy. Finally, we determine whether G acts transitively on \mathbb{S}^3 ; if so, we identify its isotropy subgroup $H \subseteq G$.*

- (a) If \tilde{g} is a round metric, i.e. $\mathcal{I}^0(\mathbb{S}^3, \tilde{g}) = SO(4) \cong SU(2)_L \times_{\mathbb{Z}_2} SU(2)_R$ and $\mathfrak{J}(\mathbb{S}^3, \tilde{g}) = \mathfrak{so}(4) \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, then the possibilities are shown in Table 1.

\mathfrak{g}	G	H (if G transitive)
$\mathfrak{so}(4)$	$SO(4)$	$SO(3)$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$SU(2) \times_{\mathbb{Z}_2} U(1) \cong U(2)$	$\text{diag}(U(1))/\mathbb{Z}_2 \cong U(1)$
$\mathfrak{u}(1) \oplus \mathfrak{su}(2)$	$U(1) \times_{\mathbb{Z}_2} SU(2) \cong U(2)$	$\text{diag}(U(1))/\mathbb{Z}_2 \cong U(1)$
$\mathfrak{su}(2)_L$	$SU(2)_L$	1
$\mathfrak{su}(2)_R$	$SU(2)_R$	1
$\text{diag}(\mathfrak{su}(2))$	$\text{diag}(SU(2))/\mathbb{Z}_2 \cong SO(3)$	G not transitive*

* In this case, G acts by rotations of $\mathbb{S}^3 \subset \mathbb{R}^4$ that fix an axis.

TABLE 1. Possibilities for \mathfrak{g} , G , and H if \tilde{g} is a round metric.

- (b) If \tilde{g} is a Berger metric, i.e. $\mathcal{I}^0(\mathbb{S}^3, \tilde{g}) = U(2)$ and $\mathfrak{J}(\mathbb{S}^3, \tilde{g}) = \mathfrak{u}(2)$, then the possibilities are shown in Table 2.

\mathfrak{g}	G	H (if G transitive)
$\mathfrak{u}(2)$	$U(2)$	$U(1)$
$\mathfrak{su}(2)$	$SU(2)$	1

TABLE 2. Possibilities for \mathfrak{g} , G , and H if \tilde{g} is a Berger metric.

- (c) If \tilde{g} is a generic left-invariant metric, i.e. $\mathcal{I}^0(\mathbb{S}^3, \tilde{g}) = SU(2)$ and $\mathfrak{J}(\mathbb{S}^3, \tilde{g}) = \mathfrak{su}(2)$, then the only possibility is $\mathfrak{g} = \mathfrak{su}(2)$ and $G = SU(2)$. In this case, G acts transitively on \mathbb{S}^3 with trivial isotropy subgroup.

Corollary 4.4. Using Corollary 4.2, we immediately obtain an analog of Lemma 4.3 for homogeneous metrics g on \mathbb{RP}^3 :

- In each case for g (round, Berger, generic left-invariant), we can classify all Lie subalgebras \mathfrak{g} of $\mathfrak{J}(\mathbb{RP}^3, g)$ with $\dim \mathfrak{g} \geq 3$ up to conjugacy in $\mathcal{I}^0(\mathbb{RP}^3, g)$. The list is exactly the same as that given in Lemma 4.3.
- For each \mathfrak{g} , we can identify the unique connected Lie subgroup $G \subset \mathcal{I}^0(\mathbb{RP}^3, g)$. These are simply the G 's from Lemma 4.3 quotiented by central \mathbb{Z}_2 's. This gives a complete list of connected Lie subgroups $G \subseteq \mathcal{I}^0(\mathbb{RP}^3, g)$ with $\dim G \geq 3$, up to conjugacy.
- All G 's act transitively on \mathbb{RP}^3 , except for the one corresponding to the last item in Table 1. For those G , the isotropy subgroup H is the image of the corresponding isotropy subgroup from Lemma 4.3 under the quotient map $G \rightarrow G/\mathbb{Z}_2$ from the previous point.

Corollary 4.5. Let g be a homogeneous metric on \mathbb{RP}^3 with $\dim \mathcal{I}(\mathbb{RP}^3, g) \geq 4$. Then any Lie subgroup $G \subseteq \mathcal{I}^0(\mathbb{RP}^3, g)$ with $\dim G \geq 4$ must act transitively.

Proof. By Corollary 4.2 and the assumption that $\dim \mathcal{I}(\mathbb{RP}^3, g) \geq 4$, g must be the standard \mathbb{Z}_2 -quotient of a round metric or Berger metric on \mathbb{S}^3 .

- If g is round, then $\mathcal{I}^0(\mathbb{RP}^3, g) = SO(4)/\mathbb{Z}_2$. By Corollary 4.4, any Lie subalgebra of $\mathfrak{J}(\mathbb{RP}^3, g) = \mathfrak{so}(4)$ with dimension at least 4 must be $\mathfrak{so}(4)$, $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, or $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. In the first case, the Killing fields clearly span the tangent space everywhere. In the

second (resp. third) case, the corresponding collection of Killing fields contains those generating left (resp. right) multiplication in $SU(2)$, so they span the tangent space at every point.

- If g is a Berger metric, then $\mathcal{I}^0(\mathbb{RP}^3, g) = U(2)/\mathbb{Z}_2$, which has dimension 4. Thus, $G = \mathcal{I}^0(\mathbb{RP}^3, g)$, which acts transitively. \square

Corollary 4.6. *If g is a metric on \mathbb{RP}^3 with $\dim \mathcal{I}(\mathbb{RP}^3, g) \geq 3$, then every connected Lie subgroup G of $\mathcal{I}^0(\mathbb{RP}^3, g)$ with dimension ≥ 3 is compact.*

Proof. If $\dim \mathcal{I}(\mathbb{RP}^3, g) = 3$ this is trivial, so we assume $\dim \mathcal{I}(\mathbb{RP}^3, g) \geq 4$. It is well-known that this implies g is homogeneous [28, p. 48–49]. Corollary 4.4 classifies all connected Lie subgroups G of $\mathcal{I}^0(\mathbb{RP}^3, g)$ with $\dim G \geq 3$. A case-by-case verification confirms that any such G is compact. \square

4.2. Proof of Theorem 1.2: dividing into cases. For the rest of this section, (M^4, g) is a complete 4-dimensional Ricci-flat ALE space with group at infinity \mathbb{Z}_2 and $\dim \ker_{L^2}(\Delta_L) \leq 3$. We now proceed to prove Theorem 1.2.

By Proposition 3.6 and the remarks after it, there is a Lie subalgebra \mathfrak{g} of Killing fields on (M, g) with $\dim \mathfrak{g} \geq 3$, which for large ρ restricts to a Lie algebra \mathfrak{g}_ρ of Killing fields on the CMC leaves (\mathbb{RP}^3, g_ρ) . These Lie algebras integrate to Lie groups $G \subseteq \mathcal{I}^0(M, g)$ and $G_\rho \subseteq \mathcal{I}^0(\mathbb{RP}^3, g_\rho)$ of dimension ≥ 3 .

By Corollary 4.6, G_ρ is compact. By Lemma 3.8, G acts smoothly, isometrically, and effectively on (M, g) , and the restriction map $G \rightarrow G_\rho$ is an isomorphism. In particular, G_ρ for all large ρ (say $\rho \geq R_0$) are isomorphic. Consider two cases:

- (A) G acts transitively on (\mathbb{RP}^3, g_ρ) for some $\rho \geq R_0$.
- (B) G does not act transitively on (\mathbb{RP}^3, g_ρ) for all $\rho \geq R_0$.

In §4.3, we will show that Case (A) implies (M, g) is either the Eguchi–Hanson space or the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$. In §4.4, we will show that Case (B) necessarily leads to (M, g) being the flat $\mathbb{R}^4/\mathbb{Z}_2$. Altogether, this proves Theorem 1.2.

4.3. Case (A). In this subsection, we assume G acts transitively on (\mathbb{RP}^3, g_ρ) , so g_ρ is homogeneous. Corollary 4.2 determines all possibilities for g_ρ and $\mathcal{I}^0(\mathbb{RP}^3, g_\rho)$. Based on this, Corollary 4.4 allows us to classify G (up to isomorphism) and the isotropy subgroup H . All possibilities are listed in Table 3.

g_ρ	$\mathcal{I}^0(\mathbb{RP}^3, g_\rho)$	G	H
Round	$SO(4)/\mathbb{Z}_2$	$SO(4)/\mathbb{Z}_2$	$SO(3)$
		$U(2)/\mathbb{Z}_2$	$U(1)$
		$SU(2)/\mathbb{Z}_2$	1
Berger	$U(2)/\mathbb{Z}_2$	$U(2)/\mathbb{Z}_2$	$U(1)$
		$SU(2)/\mathbb{Z}_2$	1
Generic left-invariant	$SU(2)/\mathbb{Z}_2$	$SU(2)/\mathbb{Z}_2$	1

TABLE 3. All possibilities for g_ρ , $\mathcal{I}^0(\mathbb{RP}^3, g_\rho)$, G , H in Case (A).

By assumption, there exists a pair (G, H) from Table 3 such that G acts smoothly, effectively, and isometrically on (M, g) with cohomogeneity one and principal isotropy H . To simplify things, we will modify G and H by using their preimages under the \mathbb{Z}_2 -quotients.

Thus, for some pair (G, H) in Table 4, G acts smoothly, *almost* effectively, and isometrically with cohomogeneity one on (M, g) and principal isotropy H .

G	H
$SO(4)$	$O(3)$
$U(2)$	$U(1) \times \mathbb{Z}_2$
$SU(2)$	\mathbb{Z}_2

TABLE 4. In Case (A), some G here acts smoothly, almost effectively, and isometrically with cohomogeneity one on (M, g) and principal isotropy H .

By Proposition 3.10, it holds that for some (G, H) from Table 4,

- (i) Outside a submanifold X , we can identify

$$(M \setminus X, g) = ((0, \infty) \times G/H, ds^2 + g(s))$$

where $\{g(s)\}_{s>0}$ are G -invariant metrics on a fixed copy of $G/H \cong \mathbb{RP}^3$.

- (ii) There is a closed subgroup K of G containing H such that X is diffeomorphic to G/K , and K/H is diffeomorphic to a spherical space form of dimension $\ell \geq 0$. Moreover, M is a smooth manifold if and only if K/H is a sphere.
- (iii) From the description of (M, g) as a bundle over X , the metric spaces $(G/H, g(s))$ converge in the Hausdorff sense to X as $s \rightarrow 0$.

We will use (ii) and (iii) with the following lemma, which follows from standard facts about closed subgroups of $SO(4)$, $U(2)$, and $SU(2)$. Below, $L(4, 1)$ is the lens space obtained by quotienting \mathbb{S}^3 by a Hopf action of \mathbb{Z}_4 .

Lemma 4.7. *For each (G, H) in Table 4, we indicate in Table 5 all closed subgroups K of G such that (i) K contains H , and (ii) K/H is diffeomorphic to a spherical space form of dimension $\ell \geq 0$. We also indicate the diffeomorphism type of G/K .*

G	H	K	G/K
$SO(4)$	$O(3)$	$SO(4)$	$\{point\}$
$U(2)$	$U(1) \times \mathbb{Z}_2$	$U(1) \times \mathbb{Z}_4, U(1) \times U(1)$	$L(4, 1), \mathbb{S}^2$
$SU(2)$	\mathbb{Z}_2	$\mathbb{Z}_4, U(1), SU(2)$	$L(4, 1), \mathbb{S}^2, \{point\}$

TABLE 5. Results for Lemma 4.7.

Since the metrics $g(s)$ are homogeneous, Corollary 4.2 implies that g is the \mathbb{Z}_2 -quotient of a metric of the form

$$\tilde{g} = ds^2 + \lambda(s)^2 \sigma_1(s) \otimes \sigma_1(s) + \mu(s)^2 \sigma_2(s) \otimes \sigma_2(s) + \nu(s)^2 \sigma_3(s) \otimes \sigma_3(s)$$

on $(0, \infty) \times SU(2)$, where λ, μ, ν are smooth positive functions, and for each $s > 0$, $\{\sigma_1(s), \sigma_2(s), \sigma_3(s)\}$ is a left-invariant coframe on $SU(2)$. It is well-known (see e.g. [16, §3] for a proof) that Ricci-flatness leads to diagonalizability of such metrics, i.e. $\sigma_i(s)$ can be chosen independently of s . Thus,

$$g(s) = \lambda(s)^2 \sigma_1 \otimes \sigma_1 + \mu(s)^2 \sigma_2 \otimes \sigma_2 + \nu(s)^2 \sigma_3 \otimes \sigma_3.$$

From this, we see that any Hausdorff limit of $(\mathbb{R}\mathbb{P}^3, g(s))$ as $s \rightarrow 0$ is homeomorphic to either $\mathbb{R}\mathbb{P}^3$ or a space of lower dimension. This rules out $G/K \cong L(4, 1)$, so Lemma 4.7 leaves us with two possibilities: either $G/K \cong \mathbb{S}^2$ or G/K is a point.

- If G/K is a point, then M is diffeomorphic to the orbifold $\mathbb{R}^4/\mathbb{Z}_2$. Its universal orbifold cover is $(\mathbb{R}^4, \tilde{g})$ where \tilde{g} is a Ricci-flat, asymptotically Euclidean metric. By the sharp case of the Bishop–Gromov volume comparison theorem, \tilde{g} is the Euclidean metric. Hence g is flat $\mathbb{R}^4/\mathbb{Z}_2$.
- If $G/K = \mathbb{S}^2$, then M is a smooth manifold diffeomorphic to $\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-2) \cong T^*\mathbb{S}^2$. By a theorem of Lock and Viaclovsky [35, Theorem 1.5], (M, g) is hyperkähler. Kronheimer’s classification [32] now implies that (M, g) is Eguchi–Hanson space.

This concludes Case (A).

4.4. Case (B). In this subsection, we assume G does not act transitively on $(\mathbb{R}\mathbb{P}^3, g_\rho)$ for all $\rho \geq R_0$. It will be shown that (M, g) must be the flat $\mathbb{R}^4/\mathbb{Z}_2$.

Corollary 4.8. *We have $\dim G = 3$.*

Proof. We already know that $\dim G \geq 3$. If $\dim G \geq 4$, then the isometry group of $(\mathbb{R}\mathbb{P}^3, g_\rho)$ for each $\rho \geq R_0$ has dimension at least 4. This makes g_ρ a homogeneous metric. By Corollary 4.5, G must act transitively on $(\mathbb{R}\mathbb{P}^3, g_\rho)$, a contradiction. \square

Let $(\tilde{M}_\infty, \tilde{g})$ be the Riemannian universal cover of the end of (M, g) . Then $(\tilde{M}_\infty, \tilde{g})$ is an incomplete asymptotically Euclidean manifold of order 4, and is foliated by CMC 3-spheres. By lifting the action of G on the end of (M, g) , we obtain a compact covering group \tilde{G} acting isometrically and effectively on $(\tilde{M}_\infty, \tilde{g})$. By Corollary 4.8, $\dim \tilde{G} = 3$. This restricts to an isometric and effective action on all CMC leaves $(\mathbb{S}^3, \tilde{g}_\rho)$, so \tilde{G} can be identified as a subgroup of $\mathcal{I}(\mathbb{S}^3, \tilde{g}_\rho)$. Its action on $(\mathbb{S}^3, \tilde{g}_\rho)$ is not transitive. Let \tilde{G}^0 be its identity component. By the classification of compact connected Lie groups of dimension 3, \tilde{G}^0 is $SU(2)$, $SO(3)$, or \mathbb{T}^3 . However, we can rule out \mathbb{T}^3 as it cannot act effectively on $(\mathbb{S}^3, \tilde{g}_\rho)$.

Next we determine the cohomogeneity of the action of \tilde{G}^0 on $(\tilde{M}_\infty, \tilde{g})$, which is one plus the cohomogeneity of the action restricted to $(\mathbb{S}^3, \tilde{g}_\rho)$. On $(\mathbb{S}^3, \tilde{g}_\rho)$, the principal orbits have positive dimension as the action is effective, but they cannot have dimension 3 as the action is not transitive. Neither can they have dimension 1, because $SO(3)$ and $SU(2)$ do not contain closed subgroups of dimension 2. Therefore, the principal orbits have dimension 2, and the action has cohomogeneity one on $(\mathbb{S}^3, \tilde{g}_\rho)$. This also rules out $\tilde{G}^0 = SU(2)$, as $SU(2)$ cannot act effectively with cohomogeneity one on a 3-sphere (see e.g. [25]). Thus, $\tilde{G}^0 = SO(3)$. To summarize:

Proposition 4.9. *$\tilde{G}^0 \cong SO(3)$ acts smoothly, isometrically, and effectively on $(\tilde{M}_\infty, \tilde{g})$ with cohomogeneity two. The action preserves all CMC leaves $(\mathbb{S}^3, \tilde{g}_\rho)$, and restricts to act with cohomogeneity one on each leaf.*

Corollary 4.10. (a) *The action has principal isotropy subgroup $H \cong SO(2)$.*

(b) *Each principal orbit is diffeomorphic to \mathbb{S}^2 and the metric on it is round.*

(c) *The induced metric on each CMC leaf $(\mathbb{S}^3, \tilde{g}_\rho)$ is of the form*

$$ds^2 + f(s)^2 g_{\mathbb{S}^2},$$

where $s \in (0, 1)$, and f is a smooth positive function with $f(0) = f(1) = 0$.

Proof. Since the principal orbits are 2-dimensional, the principal isotropy subgroup H is a 1-dimensional closed Lie subgroup of $SO(3)$. The only ones up to conjugacy are $SO(2)$

and $O(2)$. If $H \cong O(2)$, then each principal orbit is an $SO(3)/O(2) \cong \mathbb{RP}^2$ embedded in a CMC leaf of \tilde{M}_∞ . This is a contradiction as \mathbb{RP}^2 cannot embed into \mathbb{S}^3 . Hence, $H \cong SO(2)$ and part (a) follows. The principal orbits are then $SO(3)/SO(2) \cong \mathbb{S}^2$ equipped with $SO(3)$ -invariant metrics, which are necessarily round. This proves (b). Finally, (c) follows by restricting to a CMC leaf where the action has cohomogeneity one, then applying the structure theory of cohomogeneity one metrics (e.g. [1, §6.3]). \square

Let $\tilde{M}_\infty^{\text{prin}} \subset \tilde{M}_\infty$ be the union of principal orbits. The orbit space $B := \tilde{M}_\infty^{\text{prin}}/\tilde{G}^0$ is a connected Riemannian manifold [1, Theorem 3.82]. By Corollary 4.10 and our setup, B is clearly diffeomorphic to \mathbb{R}^2 . For each $x \in \tilde{M}_\infty^{\text{prin}}$, we have

$$T_x \tilde{M}_\infty^{\text{prin}} = \mathcal{H}_x \oplus \mathcal{V}_x,$$

where \mathcal{V}_x is the tangent space to the \mathbb{S}^2 orbit and $\mathcal{H}_x = \mathcal{V}_x^\perp$ is its \tilde{g} -orthogonal complement. Since \tilde{G}^0 acts isometrically, it preserves the subbundles \mathcal{H} and \mathcal{V} . Restricting to the isotropy subgroup H at x gives the following.

- \mathcal{H}_x carries the slice representation of $H = SO(2)$. This is a trivial representation because the orbit of x is principal [1, §3.4].
- \mathcal{V}_x carries the isotropy representation of H , which is identified with the adjoint representation of H on an $\text{Ad}(H)$ -invariant complement of \mathfrak{h} in \mathfrak{g} . As one checks, this is isomorphic to the defining representation of $SO(2)$.

Let $(\cdot)^\mathcal{H}$ and $(\cdot)^\mathcal{V}$ denote orthogonal projections of tangent vectors to \mathcal{H} and \mathcal{V} respectively. Following O'Neill [39], we define tensor fields A and T by setting at each $x \in \tilde{M}_\infty^{\text{prin}}$

$$\begin{aligned} A_x : \wedge^2 \mathcal{H}_x &\rightarrow \mathcal{V}_x, & A_x(X, Y) &= \frac{1}{2}[\tilde{X}, \tilde{Y}]^\mathcal{V}, \\ T_x : \mathcal{V}_x \otimes \mathcal{V}_x &\rightarrow \mathcal{H}_x, & T_x(Z, W) &= (\nabla_{\tilde{Z}} \tilde{W})^\mathcal{H}, \end{aligned}$$

where \tilde{X} and \tilde{Y} are local horizontal extensions of X and Y , and \tilde{Z} and \tilde{W} are local vertical extensions of Z and W . Vanishing of A is equivalent to integrability of the horizontal distribution. Meanwhile, T is the second fundamental form of the \mathbb{S}^2 orbits. It is straightforward to verify that for all $\phi \in \tilde{G}^0$,

$$(4.3) \quad A(d\phi(X), d\phi(Y)) = d\phi(A(X, Y)), \quad T(d\phi(Z), d\phi(W)) = d\phi(T(Z, W)).$$

In particular, A_x and T_x are H -equivariant maps at each $x \in \tilde{M}_\infty^{\text{prin}}$.

Lemma 4.11. (a) *The horizontal distribution $\mathcal{H} \subset T\tilde{M}_\infty^{\text{prin}}$ is integrable.*

- (b) *Each principal orbit is umbilic, i.e. $T_x(Z, W) = \tilde{g}_x(Z, W)\mathbf{H}_x$ for all $Z, W \in \mathcal{V}_x$ where $\mathbf{H}_x \in \mathcal{H}_x$ is the mean curvature vector.*
- (c) *Each principal orbit is spheric, i.e. the mean curvature vector field \mathbf{H} along any principal orbit satisfies $(\nabla_Z \mathbf{H})^\mathcal{H} = 0$ for all vertical vectors $Z \in \mathcal{V}_x$.*

Proof. (a) Since \mathcal{H}_x is a trivial H -module, so is $\wedge^2 \mathcal{H}_x$. Also, \mathcal{V}_x has no nonzero vectors fixed by H . Since A_x is H -equivariant, these facts force $A_x = 0$.

(b) Since T_x is H -equivariant, and \mathcal{H}_x is a trivial H -module, it follows that T_x is H -invariant. Take a basis $\{e_1, e_2\}$ for \mathcal{H}_x and write $T_x(Z, W) = T_x^{(1)}(Z, W)e_1 + T_x^{(2)}(Z, W)e_2$; then $T_x^{(i)} : \mathcal{V}_x \otimes \mathcal{V}_x \rightarrow \mathbb{R}$ is H -invariant for $i = 1, 2$. By Schur's lemma, there is a one-dimensional space of H -invariant symmetric 2-tensors on \mathcal{V}_x spanned by \tilde{g}_x . Hence, $T_x^{(1)}$ and $T_x^{(2)}$ each are multiples of \tilde{g}_x , varying smoothly in x . Thus, $T_x(Z, W) = \tilde{g}_x(Z, W)\nu_x$ for some $\nu_x \in \mathcal{H}_x$. As T_x is the second fundamental form of the orbit at x , we must have $\nu_x = \mathbf{H}_x$.

(c) Using part (b) and (4.3), we see that $d\phi(\mathbf{H}_x) = \mathbf{H}_{\phi(x)}$ for each $\phi \in \tilde{G}^0$. Then one verifies that

$$(4.4) \quad d\phi((\nabla_Z \mathbf{H})^{\mathcal{H}}) = (\nabla_{d\phi(Z)} \mathbf{H})^{\mathcal{H}} \quad \text{for all } \phi \in \tilde{G}^0, Z \in \mathcal{V}_x.$$

If $\phi \in H$, then (4.4) implies that $(\nabla \mathbf{H})^{\mathcal{H}} : \mathcal{V}_x \rightarrow \mathcal{H}_x$ is H -equivariant. Again, by the triviality of \mathcal{H}_x and the fact that \mathcal{V}_x has no points fixed by H , this yields $(\nabla \mathbf{H})^{\mathcal{H}} = 0$ at x . Then (4.4) allows us to transport this across the orbit, proving the claim. \square

Proposition 4.12. *$(\tilde{M}_\infty^{\text{prin}}, \tilde{g})$ is isometric to a warped product $(B \times \mathbb{S}^2, h + \xi^2 g_{\mathbb{S}^2})$, where h is a metric on $B \cong \mathbb{R}^2$ and $\xi : B \rightarrow (0, \infty)$ is a smooth positive function.*

Proof. By [1, Remark 3.99] and Corollary 4.10, the quotient map $\pi : \tilde{M}_\infty^{\text{prin}} \rightarrow B$ is a Riemannian submersion with round \mathbb{S}^2 fibers. Fix a basepoint $b_0 \in B$ and identify $\pi^{-1}(b_0)$ with \mathbb{S}^2 . For any $b \in B$, any smooth path $\gamma : [0, 1] \rightarrow B$ with $\gamma(0) = b_0$ and $\gamma(1) = b$, and any $\theta \in \mathbb{S}^2$, let $\tilde{\gamma} : [0, 1] \rightarrow \tilde{M}_\infty^{\text{prin}}$ be the horizontal lift of γ starting at θ . Horizontal transport along this curve defines a diffeomorphism $P_\gamma : \mathbb{S}^2 \cong \pi^{-1}(b_0) \rightarrow \pi^{-1}(b)$. Since the horizontal distribution \mathcal{H} is integrable by Lemma 4.11(a), P_γ only depends on the homotopy class of γ . But $B \cong \mathbb{R}^2$ is contractible, so P_γ is independent of the path γ joining b_0 to b . Therefore, we have a well-defined smooth map $P_b : \mathbb{S}^2 \rightarrow \pi^{-1}(b)$. A global diffeomorphism is given by

$$B \times \mathbb{S}^2 \rightarrow \tilde{M}_\infty^{\text{prin}}, \quad (b, \theta) \mapsto P_b(\theta).$$

Under this diffeomorphism, the horizontal distribution \mathcal{H} is identified with TB and the vertical distribution \mathcal{V} is identified with $T\mathbb{S}^2$. Integrability of \mathcal{H} implies that the submanifold $B \times \{\theta\} \subset B \times \mathbb{S}^2$ is totally geodesic for each $\theta \in \mathbb{S}^2$. Moreover, for each $b' \in B$, the submanifold $\{b'\} \times \mathbb{S}^2$ corresponds to a fiber of π , which by Lemma 4.11 is spheric. We now apply [41, Proposition 3(c)] to conclude that $(\tilde{M}_\infty^{\text{prin}}, \tilde{g})$ is a warped product. \square

Lemma 4.13. *There is an asymptotically Euclidean diffeomorphism $\Phi : \tilde{M}_\infty \rightarrow \mathbb{R}^4 \setminus B_R(0)$ (as in Definition 1.1 with $\Gamma = \{1\}$) and a constant $A > 0$ with the following properties.*

- (a) Let $\mathcal{C} = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 \leq w^2\}$ and $\Omega := (\mathbb{R}^4 \setminus B_R(0)) \setminus \mathcal{C}$. Then $\Phi^{-1}(\Omega) \subset \tilde{M}_\infty^{\text{prin}}$.
- (b) Let \tilde{r} be the Riemannian distance from a point with respect to \tilde{g} . On $\Phi^{-1}(\Omega)$, we have $A^{-1}\tilde{r} \leq \xi \leq A\tilde{r}$ and $|d\xi| = 1 + \mathcal{O}(\tilde{r}^{-4})$.
- (c) View $\Phi^{-1}(\Omega) \subset \tilde{M}_\infty^{\text{prin}}$ as a subset of $B \times \mathbb{S}^2$ under the identification from Proposition 4.12. Then $\Phi^{-1}(\Omega)$ contains an open set of the form $\Omega_B \times \mathbb{S}^2$, where $\Omega_B \subset B$ is an open neighborhood of a flow half-line of $\nabla \xi$.

Proof. By Theorem 3.2, there is an ALE diffeomorphism $M \setminus K \rightarrow (\mathbb{R}^4 \setminus B_R(0))/\mathbb{Z}_2$ which maps CMC hypersurfaces to coordinate spheres (mod \mathbb{Z}_2). Lifting this gives an asymptotically Euclidean diffeomorphism $\Phi : \tilde{M}_\infty \rightarrow \mathbb{R}^4 \setminus B_R(0)$ mapping CMC 3-spheres to coordinate spheres. Viewed in $\mathbb{R}^4 \setminus B_R(0)$, the action of \tilde{G}^0 on \tilde{M}_∞ is

$$(4.5) \quad \Psi : \tilde{G}^0 \times (\mathbb{R}^4 \setminus B_R(0)) \rightarrow \mathbb{R}^4 \setminus B_R(0), \quad \Psi(\phi, x) = \Phi(\phi \cdot \Phi^{-1}(x)).$$

For each $\rho > R$, we also define (viewing \mathbb{S}^3 as the unit sphere in \mathbb{R}^4)

$$(4.6) \quad \Psi_\rho : \tilde{G}^0 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3, \quad \Psi_\rho(\phi, \theta) = \rho^{-1}\Psi(\phi, \rho\theta).$$

Let X_1, X_2, X_3 be fundamental vector fields generating the action (4.5) and satisfying

$$(4.7) \quad [X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

Then

$$Y_{i,\rho}(\theta) = \rho^{-1}X_i(\rho\theta), \quad i = 1, 2, 3$$

are fundamental vector fields generating (4.6) and satisfy the same Lie algebra relations. By Proposition 3.6(e), there exist linearly independent Killing fields $Y_{i,\infty}$, $i = 1, 2, 3$ on the round $(\mathbb{S}^3, \bar{g}_\infty)$ such that

$$(4.8) \quad \|Y_{i,\rho} - Y_{i,\infty}\|_{C^k(\bar{g}_\infty)} \leq \mathcal{O}(\rho^{-3+\varepsilon})$$

for all $k \geq 0$. In particular, $Y_{i,\infty}$ satisfy the same Lie algebra relations as (4.7), so they generate a Lie subalgebra \mathfrak{g}_∞ of $\mathfrak{J}(\mathbb{S}^3, \bar{g}_\infty) = \mathfrak{so}(4)$ isomorphic to $\mathfrak{so}(3)$. The convergence (4.8), combined with standard Grönwall-type arguments, implies that the maps $\{\Psi_\rho\}_{\rho>R}$ are uniformly Cauchy in each C^k norm. Hence, there is a smooth effective action $\Psi_\infty : \tilde{G}^0 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3$ such that for each $k \geq 0$,

$$(4.9) \quad \Psi_\rho \rightarrow \Psi_\infty \text{ in } C^k(\tilde{G}^0 \times \mathbb{S}^3).$$

Moreover, Ψ_∞ acts by isometries with respect to \bar{g}_∞ .

Therefore, we are in the final case of Table 1. As noted there, this implies that the action Ψ_∞ acts by rotations fixing an axis. Thus, by (4.9), the action Ψ from (4.5) smoothly asymptotes to Euclidean rotations around an axis. By composing Φ with a Euclidean rotation, we may assume the axis of rotation is the w -axis, i.e. it is spanned by $(0, 0, 0, 1) \in \mathbb{R}^4$. See Figure 1 for an illustration.

Now let \mathcal{C} be the solid cone

$$\mathcal{C} = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 \leq w^2\}.$$

Note that all singular orbits of Ψ_∞ lie on the w -axis. Therefore, outside a sufficiently large ball, all singular orbits of Ψ are contained in a uniform tubular neighborhood of the w -axis, hence in \mathcal{C} . Thus, $\Omega' := (\mathbb{R}^4 \setminus B_R(0)) \setminus \mathcal{C}$ contains only principal orbits. This establishes part (a) of the lemma.

For each $p = (x, y, z, w) \in \mathbb{R}^4 \setminus B_R(0)$, the orbit of p under Ψ is a round \mathbb{S}^2 of radius $\xi_\infty(p) = \sqrt{x^2 + y^2 + z^2}$. In particular, along any outward Euclidean ray not parallel to the w -axis, the orbit radius ξ_∞ increases at a definite linear rate. Therefore, there exists $A > 0$ such that $A^{-1}|p| \leq \xi_\infty(p) \leq A|p|$ for all $x \in \Omega$. Thus, the same is true for the orbit radius function ξ of Ψ outside a sufficiently large ball. This proves the first claim of part (b). As for the second claim, we calculate the vertical and horizontal components of $\text{Ric} = 0$ on $(B \times \mathbb{S}^2 \cong \tilde{M}_\infty^{\text{prin}}, \tilde{g} = h + \xi^2 g_{\mathbb{S}^2})$ to be

$$(4.10) \quad \text{Ric}_h = 2\xi^{-1}(\nabla^h)^2\xi,$$

$$(4.11) \quad \xi\Delta^h\xi + |d\xi|^2 = 1.$$

(See e.g. [8, Proposition 9.106].) Since B is 2-dimensional, tracing (4.10) gives $\Delta^h\xi = \kappa_h\xi$, where κ_h is the Gauss curvature of h . Plugging this into (4.11) yields

$$|d\xi|^2 = 1 - \kappa_h\xi^2.$$

Note that $|\kappa_h| \leq C|\text{Rm}_{\tilde{g}}|_{\tilde{g}} = \mathcal{O}(\tilde{r}^{-6})$ by Theorem 3.1. Therefore, going back to Ω , and using the fact that $A^{-1}\tilde{r} \leq \xi \leq A\tilde{r}$, we obtain

$$|d\xi|^2 = 1 + \mathcal{O}(\tilde{r}^{-4}),$$

which proves (b).

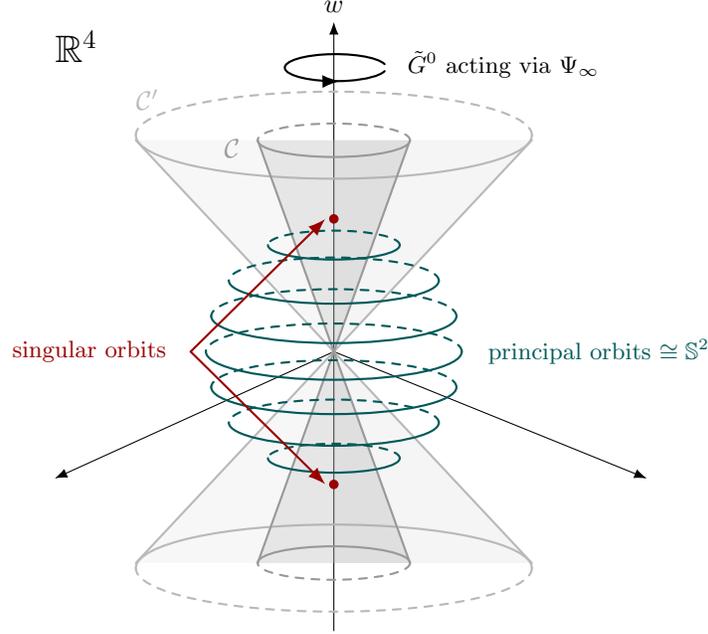


FIGURE 1. Limiting action Ψ_∞ of $\tilde{G}^0 \cong SO(3)$ on \mathbb{R}^4 , viewed with w vertical and x, y, z in the horizontal plane. The limiting action is by rotations fixing the w -axis; the actual action Ψ of \tilde{G}^0 on \tilde{M}_∞ is asymptotic to this. Examples of orbits, as well as the cones $\mathcal{C} = \{x^2 + y^2 + z^2 \leq w^2\}$ and $\mathcal{C}' = \{x^2 + y^2 + z^2 \leq 3w^2\}$ used in the proof of Lemma 4.13, are shown.

Finally, let $\Omega' = (\mathbb{R}^4 \setminus B_R(0)) \setminus \mathcal{C}' \subset \Omega$, where \mathcal{C}' is the widened cone

$$\mathcal{C}' = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 \leq 3w^2\}.$$

Observe that Ω' contains \mathbb{S}^2 orbits which can be chosen arbitrarily far from the origin. Take one of these orbits, and flow it forwards by the vector field $\nabla\xi$, which behaves like the Euclidean radial vector field near infinity. If the orbit was chosen sufficiently far from the origin, then its image under the flow never exits Ω . Part (c) follows from these observations. \square

Lemma 4.13 allows us to use ξ as a coordinate on $\Phi^{-1}(\Omega) \subset B \times \mathbb{S}^2$. Since ξ is constant on \mathbb{S}^2 , it descends to a coordinate on Ω_B . We now complete this to a full coordinate system using well-known arguments from the study of spherically symmetric spacetimes.

Proposition 4.14. *There is a domain $\Omega'_B \subset B$ which contains a flow half-line of $\nabla\xi$, and admits a coordinate system $(\xi, t) : \Omega'_B \rightarrow \mathbb{R}^2$ mapping onto an open neighborhood $D \subset \mathbb{R}^2$ of $(R, \infty) \times \{0\}$ (for some large R). In this coordinate system, the metric h can be written*

$$h = E(\xi)d\xi^2 + F(\xi)dt^2$$

for some smooth positive functions E and F .

Proof. By Proposition 4.12, Ricci-flatness, and a version of Birkhoff's theorem (e.g. [2, Proposition 2.7]), $K := (\star_h d\xi)^\sharp$ is a Killing field on (B, h) satisfying $\langle K, \nabla\xi \rangle = 0$. Since $d\xi$ is nonvanishing on the set $\Omega_B \subset B$ from Lemma 4.13, so is K .

By Lemma 4.13(c), Ω_B contains a flow half-line γ of $\nabla\xi$. Around each point of γ , we can use the flow of K to construct a local coordinate t such that $K = \partial_t$ and $\nabla\xi(t) = 0$. If two such coordinates t_α, t_β are defined on overlapping patches, then $u = t_\alpha - t_\beta$ satisfies $Ku = \nabla\xi(u) = 0$ on the overlap, which we may assume to be simply connected. Therefore, u is a constant, so by adding a constant to t_β , it can be made to agree with t_α on the overlap. Therefore, there exists a coordinate system (ξ, t) for a neighborhood Ω'_B of γ which maps to a neighborhood of $(R, \infty) \times \{0\}$ in \mathbb{R}^2 (for some large R). Since $\langle \partial_\xi, \partial_t \rangle = \langle \frac{\nabla\xi}{|\nabla\xi|^2}, K \rangle = 0$, we can write

$$h = E(\xi, t)d\xi^2 + F(\xi, t)dt^2$$

for some positive functions E and F . Since K is a Killing field, E and F are independent of t . \square

Proposition 4.15. *There is a nonempty open set $U \subset \tilde{M}_\infty^{\text{prin}}$ on which \tilde{g} is flat.*

Proof. Propositions 4.12 and 4.14 imply that

$$(4.12) \quad \tilde{g} = h + \xi^2 g_{\mathbb{S}^2} = E(\xi)d\xi^2 + F(\xi)dt^2 + \xi^2 g_{\mathbb{S}^2} \quad \text{on } D \times \mathbb{S}^2,$$

where $D(\xi, t) \subset \mathbb{R}^2$ is an open neighborhood of $(R, \infty) \times \{0\}$. Rudimentary computations yield

$$(4.13) \quad ((\nabla^h)^2 \xi)_{\xi\xi} = -\frac{E'}{2E}, \quad ((\nabla^h)^2 \xi)_{tt} = \frac{F'}{2E}.$$

Meanwhile, (4.10) gives

$$(\nabla^h)^2 \xi = \frac{\kappa_h \xi}{2} h,$$

where κ_h is the Gauss curvature of h . Combining this with (4.13) and then isolating κ_h in both equations, we obtain

$$\frac{F'}{EF} = \kappa_h = -\frac{E'}{E^2}.$$

This implies $(\log(EF))' = 0$, so $E = \frac{c}{F}$ for some $c > 0$. Thus, additionally defining $V(\xi) = F(\xi)/c$ and $\tau = \sqrt{ct}$, (4.12) becomes

$$\tilde{g} = \frac{1}{V(\xi)} d\xi^2 + V(\xi) d\tau^2 + \xi^2 g_{\mathbb{S}^2} \quad \text{on } D' \times \mathbb{S}^2,$$

where $D'(\xi, \tau) \subset \mathbb{R}^2$ is an open neighborhood of $(R, \infty) \times \{0\}$. Next, using [8, Proposition 9.106], along the spherical direction we have

$$(4.14) \quad \text{Ric}_{\tilde{g}}|_{T\mathbb{S}^2} = (1 - \xi \Delta^h \xi - |\nabla \xi|^2) g_{\mathbb{S}^2}.$$

Straightforward computation gives $|\nabla \xi|^2 = V$ and $\Delta^h \xi = V'$. Inserting this into (4.14), imposing $\text{Ric}_{\tilde{g}}|_{T\mathbb{S}^2} = 0$, and solving the resulting ODE gives

$$V(\xi) = 1 - \frac{2m}{\xi} \quad \text{for some } m \in \mathbb{R}.$$

Thus, \tilde{g} is the Euclidean Schwarzschild metric with mass m on $D' \times \mathbb{S}^2$. Suppose $m \neq 0$. If we fix some $\theta \in \mathbb{S}^2$ and consider the curve $\xi \mapsto (\xi, 0, \theta) \in D' \times \mathbb{S}^2$, then $|\text{Rm}_{\tilde{g}}| \asymp \xi^{-3}$ as $\xi \rightarrow \infty$ (a standard fact about Euclidean Schwarzschild metrics). By Lemma 4.13, ξ is comparable to the Riemannian distance \tilde{r} from a point with respect to \tilde{g} . Thus, $|\text{Rm}_{\tilde{g}}| \asymp \tilde{r}^{-3}$ along the same curve. However, \tilde{g} is the covering metric of an ALE end, so by Theorem 3.1

we must have $|\text{Rm}_{\bar{g}}| = \mathcal{O}(\tilde{r}^{-6})$. This is a contradiction. Therefore $m = 0$, which makes \tilde{g} flat in $D' \times \mathbb{S}^2$. \square

We may now finish off Case (B). By Proposition 4.15, the downstairs metric g is flat in an open set of M . Since Ricci-flat metrics are real-analytic, unique continuation implies that (M, g) is flat everywhere. Hence, (M, g) must be the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$. This concludes Case (B) and hence the proof of Theorem 1.2.

5. A CHARACTERIZATION OF CALABI SPACE

In this section, we prove Theorem 1.4. For the rest of this section, (M^{2m}, g, J) is a complete Ricci-flat Kähler ALE manifold of complex dimension $m \geq 3$ with group at infinity \mathbb{Z}_m , satisfying

$$\dim \ker_{L^2}(\Delta_L) \leq \begin{cases} 2m - 3 & \text{if } m \neq 4, \\ 2 & \text{if } m = 4. \end{cases}$$

We begin by laying out several facts regarding homogeneous metrics on $\mathbb{S}^{2m-1}/\mathbb{Z}_m$ and their isometry groups, which are proved in [33, §5]. Recall that $\mathbb{S}^{2m-1}/\mathbb{Z}_m$ refers to the lens space described in §2.3, i.e. where \mathbb{Z}_m acts by Hopf rotations.

Lemma 5.1 ([33, Lemma 5.3]). *If \mathfrak{g} is a Lie subalgebra of $\mathfrak{u}(m)$ satisfying*

$$\dim \mathfrak{g} \geq m^2 - 2m + 3,$$

then either $\mathfrak{g} = \mathfrak{su}(m)$ or $\mathfrak{g} = \mathfrak{u}(m)$. If $m = 4$ we also allow $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ (up to conjugacy in $\mathfrak{u}(4)$).

For the next result, recall the definition of d_1 stated before Lemma 3.9. As before, \bar{g}_∞ denotes the metric of constant sectional curvature 1.

Proposition 5.2 ([33, Proposition 5.4]). *We have $d_1(\mathbb{S}^{2m-1}/\mathbb{Z}_m, \bar{g}_\infty) \geq 2m - 3$.*

Next, we recall the quotient-Berger metrics $\{g_{a,b}\}_{a,b>0}$ on $\mathbb{S}^{2m-1}/\mathbb{Z}_m$ that were defined in §2.3. The following result characterizes such metrics on $\mathbb{S}^{2m-1}/\mathbb{Z}_m$ in terms of isometry group dimensions.

Proposition 5.3 ([33, Proposition 5.5]). *If g is a homogeneous metric on $\mathbb{S}^{2m-1}/\mathbb{Z}_m$ with*

$$\dim \mathcal{I}(\mathbb{S}^{2m-1}/\mathbb{Z}_m, g) \geq \begin{cases} m^2 - 2m + 3 & \text{if } m = 3 \text{ or } m \geq 5, \\ 14 & \text{if } m = 4, \end{cases}$$

then $g = g_{a,b}$ for some $a, b > 0$. Thus $\mathcal{I}^0(\mathbb{S}^{2m-1}/\mathbb{Z}_m, g) = U(m)/\mathbb{Z}_m$, and the isotropy subgroup of its action is the image of $U(m-1) \times \mathbb{Z}_m$ under the quotient map $U(m) \rightarrow U(m)/\mathbb{Z}_m$.

Corollary 5.4. *The group $SU(m)$ acts smoothly, almost effectively, and isometrically with cohomogeneity one on (M, g) . The principal isotropy subgroup is $S(U(m-1) \times \mathbb{Z}_m)$.*

Proof. In addition to the facts listed above, the proof of this requires Lemma 3.8, Lemma 3.9, and Proposition 3.10. Then the proof is identical to that of [33, Corollary 5.6], so we omit the details. \square

Lemma 5.5 ([33, Lemma 5.7 and Corollary 5.8]). *If K is a closed subgroup of $SU(m)$ containing $S(U(m-1) \times \mathbb{Z}_m)$, and $K/S(U(m-1) \times \mathbb{Z}_m)$ is diffeomorphic to \mathbb{S}^ℓ/Γ for some $\ell \geq 0$, then K is one of $S(U(m-1) \times \mathbb{Z}_{2m})$, $S(U(m-1) \times U(1))$, or $SU(m)$.*

Lemma 5.6 ([33, Lemma 5.9]). *Any $SU(m)$ -invariant metric on $SU(m)/S(U(m-1) \times \mathbb{Z}_m) \cong \mathbb{S}^{2m-1}/\mathbb{Z}_m$ is equal to $g_{a,b}$ for some $a, b > 0$.*

Proposition 5.7. *Either (a) or (b) holds:*

- (a) M is diffeomorphic to $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$. Moreover, identifying the complement of the zero section with $(0, \infty) \times \mathbb{S}^{2m-1}/\mathbb{Z}_m$, we have $g = ds^2 + g_{a(s),b(s)}$ where a and b are positive analytic functions with boundary conditions $a(0) = 0$, $a'(0) = m$, $b(0) > 0$, and $b'(0) = 0$.
- (b) Up to possible negation of the complex structure, (M, g, J) is biholomorphically isometric to the flat Kähler orbifold $\mathbb{C}^m/\mathbb{Z}_m$.

Proof. Let $G = SU(m)$ and $H = S(U(m-1) \times \mathbb{Z}_m)$. By Corollary 5.4 and Proposition 3.10, we get the following.

- (i) Outside a submanifold X , we can identify

$$(M \setminus X, g) = ((0, \infty) \times G/H, ds^2 + g(s)),$$

where $\{g(s)\}_{s>0}$ are G -invariant metrics on a fixed copy of $G/H \cong \mathbb{S}^{2m-1}/\mathbb{Z}_m$.

- (ii) There is a closed subgroup K of G containing H such that K/H is diffeomorphic to a spherical space form of dimension $\ell \geq 0$. Moreover, M is a smooth manifold if and only if K/H is a sphere.
- (iii) From the description of (M, g) as a bundle over X , the metric spaces $(G/H, g(s))$ converge in the Hausdorff sense to X as $s \rightarrow 0$.

From (i) and Lemma 5.6, we have $g(s) = g_{a(s),b(s)}$ for some positive functions $a, b : (0, \infty) \rightarrow (0, \infty)$. By Ricci-flatness, a and b must be analytic. By (ii) and Lemma 5.5, there are three possibilities for K and X :

- (a) $K = S(U(m-1) \times \mathbb{Z}_{2m})$ and $X \cong \mathbb{S}^{2m-1}/\mathbb{Z}_{2m}$.
- (b) $K = S(U(m-1) \times U(1))$ and $X \cong \mathbb{C}\mathbb{P}^{m-1}$.
- (c) $K = SU(m)$ and X is a point.

However, the first option is incompatible with (iii) because it is impossible for $(G/H, g_{a(s),b(s)})$ to converge in the Hausdorff sense to a metric space homeomorphic to $\mathbb{S}^{2m-1}/\mathbb{Z}_{2m}$. Indeed, any change in topology must arise from sending a or b to zero, which necessarily results in a dimension drop. Hence we are left with two possibilities:

- If $X \cong \mathbb{C}\mathbb{P}^{m-1}$, then M is diffeomorphic to $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$. For this to happen, we need $a(s) \rightarrow 0$ and $b(s) \rightarrow b(0) > 0$ as $s \rightarrow 0$. For the metric to close up smoothly at X , we adapt the discussion in [43, §3] to obtain $a'(0) = m$ and $b'(0) = 0$ as necessary conditions. This gives part (a) of the proposition.
- If X is a point, then M is diffeomorphic to the orbifold $\mathbb{C}^m/\mathbb{Z}_m$. Its universal orbifold cover is a complete Ricci-flat Kähler asymptotically Euclidean manifold $(\mathbb{C}^m, \tilde{g}, \tilde{J})$. By the sharp case of the Bishop–Gromov volume comparison theorem, \tilde{g} is the Euclidean metric. Kählerity then necessitates that $(\mathbb{C}^m, \tilde{g}, \tilde{J})$ is biholomorphically isometric to the standard \mathbb{C}^m , up to possible negation of the complex structure. This gives part (b) of the proposition.

□

In particular, Proposition 5.7 establishes the flat $\mathbb{C}^m/\mathbb{Z}_m$ case of Theorem 1.4. It remains to characterize (M, g, J) as the Calabi space in the other case. So far, the metric is of the correct form, but more needs to be said about the complex structure. We will do this by

finding J explicitly. This requires some notation. Let us first use Proposition 5.7 to write, outside the central $\mathbb{C}\mathbb{P}^{m-1}$,

$$g = ds^2 + g_{a(s),b(s)} = ds^2 + a(s)^2\sigma \otimes \sigma + b(s)^2\pi^*g_{\text{FS}},$$

where a and b are positive analytic functions on $(0, \infty)$. Write $M^{\text{prin}} = M \setminus \mathbb{C}\mathbb{P}^{m-1}$. The projection $\pi : \mathbb{S}^{2m-1}/\mathbb{Z}_m \rightarrow \mathbb{C}\mathbb{P}^{m-1}$ induces a Riemannian submersion

$$(5.1) \quad \tilde{\pi} : (M^{\text{prin}}, g) \rightarrow ((0, \infty) \times \mathbb{C}\mathbb{P}^{m-1}, ds^2 + b(s)^2g_{\text{FS}}).$$

Let ξ be the vector field algebraically dual to σ , so that $\sigma(\xi) = 1$. Then $U = \frac{1}{a(s)}\xi$ is a unit vector field tangential to the Hopf fibers. Fixing a point $p \in M^{\text{prin}}$, we may decompose

$$T_pM = \mathcal{H}_p \oplus \mathcal{V}_p,$$

where $\mathcal{H}_p = \tilde{\pi}^*(T_{\tilde{\pi}(p)}\mathbb{C}\mathbb{P}^{m-1})$ is the horizontal lift of $T_{\tilde{\pi}(p)}\mathbb{C}\mathbb{P}^{m-1}$, and $\mathcal{V}_p = \mathcal{H}_p^\perp = \text{span}\{\partial_s, U\}$ is its orthogonal complement. Despite what the notation may suggest, the horizontal subspace at p of the Riemannian submersion $\tilde{\pi}$ is $\mathcal{H}_p \oplus \text{span}\{\partial_s\}$ rather than \mathcal{H}_p itself. Likewise, the vertical space at p is $\text{span}\{U\}$ and not \mathcal{V}_p .

Below, we denote by $I = \tilde{\pi}^*J_{\text{FS}}$ the horizontal lift of the complex structure on $\mathbb{C}\mathbb{P}^{m-1}$. Note that I acts trivially on ∂_s , so it restricts to a complex structure on \mathcal{H}_p . We will prove the following:

Lemma 5.8. *At all points in a dense open subset of M^{prin} , we have $J = \varepsilon_{\mathcal{H}}I + \varepsilon_{\mathcal{V}}J_{\mathcal{V}}$ for some $\varepsilon_{\mathcal{H}}, \varepsilon_{\mathcal{V}} \in \{+1, -1\}$, where $J_{\mathcal{V}}$ is defined by*

$$J_{\mathcal{V}}(\partial_s) = U, \quad J_{\mathcal{V}}(U) = -\partial_s, \quad J_{\mathcal{V}}(\mathcal{H}_p) = 0.$$

This will be proved by computing certain curvature operators and commuting them with J . Below, for $X, Y \in T_pM$, the operator $X \wedge Y \in \mathfrak{so}(T_pM)$ is defined by $(X \wedge Y)(Z) = \langle Y, Z \rangle X - \langle X, Z \rangle Y$.

Lemma 5.9. *For each $X, Y, Z \in \mathcal{H}_p$, we have*

$$(5.2)$$

$$\text{Rm}(X, Y)\partial_s = - \left(\left(\frac{a}{b^2} \right)' + \frac{a'}{b^2} \right) \langle IX, Y \rangle U,$$

$$(5.3)$$

$$\text{Rm}(X, Y)U = \left(\left(\frac{a}{b^2} \right)' + \frac{a'}{b^2} \right) \langle IX, Y \rangle \partial_s,$$

$$(5.4)$$

$$\text{Rm}(X, Y)Z = \frac{(b')^2 - 1}{b^2} (X \wedge Y)(Z) + \left(\frac{a^2}{b^4} - \frac{1}{b^2} \right) (IX \wedge IY)(Z) + \left(\frac{2}{b^2} - \frac{2a^2}{b^4} \right) \langle IX, Y \rangle IZ.$$

In particular, $\text{Rm}(X, Y) \in \mathfrak{so}(T_pM)$ preserves the splitting $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$.

Proof. Let $X, Y, Z, W \in \mathcal{H}_p$. Applying O'Neill's formulae [39] to the Riemannian submersion $\tilde{\pi}$, we have

$$\langle \text{Rm}(X, Y)\partial_s, U \rangle = \langle (\nabla_{\partial_s} A)_X Y, U \rangle + \langle A_X Y, T_U \partial_s \rangle - \langle A_Y \partial_s, T_U X \rangle - \langle A_{\partial_s} X, T_U Y \rangle,$$

$$\langle \text{Rm}(X, Y)Z, U \rangle = \langle (\nabla_Z A)_X Y, U \rangle + \langle A_X Y, T_U Z \rangle - \langle A_Y Z, T_U X \rangle - \langle A_Z X, T_U Y \rangle,$$

$$\langle \text{Rm}(X, Y)Z, \partial_s \rangle = \langle \text{Rm}^{\tilde{\pi}}(X, Y)Z, \partial_s \rangle - 2\langle A_X Y, A_Z \partial_s \rangle + \langle A_Y Z, A_X \partial_s \rangle + \langle A_Z X, A_Y \partial_s \rangle,$$

$$\langle \text{Rm}(X, Y)Z, W \rangle = \langle \text{Rm}^{\tilde{\pi}}(X, Y)Z, W \rangle - 2\langle A_X Y, A_Z W \rangle + \langle A_Y Z, A_X W \rangle + \langle A_Z X, A_Y W \rangle,$$

where $\text{Rm}^{\tilde{\pi}}(X, Y)Z = \tilde{\pi}^*(\text{Rm}_0(\tilde{\pi}_* X, \tilde{\pi}_* Y)\tilde{\pi}_* Z)$ and Rm_0 is the curvature of the base manifold of $\tilde{\pi}$. The tensors A and T are defined as follows: for $E, F \in \mathcal{H}_p$, we arbitrarily extend them to \mathcal{H} -valued vector fields around p denoted by the same letters, and define

$$A_E F = \frac{1}{2}\langle [E, F], U \rangle U, \quad T_U E = \langle \nabla_U E, U \rangle U.$$

Meanwhile, we let

$$A_E \partial_s = \frac{1}{2}\langle [E, \partial_s], U \rangle U, \quad T_U \partial_s = \langle \nabla_U \partial_s, U \rangle U.$$

We will now compute all the A and T combinations appearing above. Below, we extend the vectors $X, Y, Z, W \in \mathcal{H}_p$ to local \mathcal{H} -valued vector fields around p by extending in the ∂_s direction with $[\partial_s, X] = [\partial_s, Y] = [\partial_s, Z] = [\partial_s, W] = 0$, then by horizontally lifting in the ξ direction (so that $[\xi, X] = [U, X] = 0$).

Firstly, since $[X, \partial_s] = 0$, we have

$$(5.5) \quad A_X \partial_s = 0.$$

Next, since $U = \frac{1}{a}\xi$ where ξ is algebraically dual to σ , and $\sigma(X) = \sigma(Y) = 0$, we have $A_X Y = \frac{a}{2}\sigma([X, Y])U = -\frac{a}{2}d\sigma(X, Y)$. Combined with the fact that $d\sigma = 2\pi^*\omega_{\text{FS}} = 2\pi^*g_{\text{FS}}(J_{\text{FS}}(\cdot), \cdot)$ for the Hopf fibration $\pi : \mathbb{S}^{2m-1}/\mathbb{Z}_m \rightarrow \mathbb{C}\mathbb{P}^{m-1}$, this implies

$$A_X Y = -\frac{a}{b^2}\langle IX, Y \rangle U$$

where we recall that $I := \tilde{\pi}^* J_{\text{FS}}$. Since $(\mathbb{C}\mathbb{P}^{m-1}, g_{\text{FS}}, J_{\text{FS}})$ is Kähler and $\tilde{\pi}$ is a Riemannian submersion, it follows that $\langle (\nabla_Z I)X, Y \rangle = 0$. Then

$$\nabla_Z (A_X Y) = -\frac{a}{b^2} [\langle I(\nabla_Z X), Y \rangle U + \langle IX, \nabla_Z Y \rangle U + \langle IX, Y \rangle \nabla_Z U],$$

which implies that

$$(\nabla_Z A)_X Y = \nabla_Z (A_X Y) - A_{\nabla_Z X} Y - A_X (\nabla_Z Y) = -\frac{a}{b^2}\langle IX, Y \rangle \nabla_Z U.$$

Likewise, since $\nabla_{\partial_s} X = \frac{b'}{b}X$ and $\nabla_{\partial_s} Y = \frac{b'}{b}Y$, we compute

$$(\nabla_{\partial_s} A)_X Y = \nabla_{\partial_s} (A_X Y) = -\left(\frac{a}{b^2}\right)' \langle IX, Y \rangle U - \frac{a}{b^2} \langle IX, Y \rangle \nabla_{\partial_s} U.$$

Next, since $[U, X] = 0$ and U has unit length, we have

$$T_U X = \langle \nabla_U X, U \rangle U = \langle \nabla_X U, U \rangle U = \frac{1}{2}X|U|^2 = 0.$$

Meanwhile, since $[\xi, \partial_s] = 0$ and $U = \frac{1}{a}\xi$, we have

$$(5.6) \quad T_U \partial_s = \frac{1}{a^2} \langle \nabla_{\xi} \partial_s, \xi \rangle U = \frac{1}{a^2} \langle \nabla_{\partial_s} \xi, \xi \rangle U = \frac{1}{2a^2} \partial_s |\xi|^2 U = \frac{a'}{a} U.$$

We take all the computations between (5.5) and (5.6), and substitute them back into O'Neill's formulas stated at the beginning of the proof. We also use $\langle \nabla_{(\cdot)} U, U \rangle = 0$ repeatedly since U is a unit vector field. This gives

$$\begin{aligned} \langle \text{Rm}(X, Y) \partial_s, U \rangle &= \left(- \left(\frac{a}{b^2} \right)' - \frac{a'}{b^2} \right) \langle IX, Y \rangle, \\ \langle \text{Rm}(X, Y) Z, U \rangle &= 0, \\ \langle \text{Rm}(X, Y) Z, \partial_s \rangle &= \langle \text{Rm}^{\tilde{\pi}}(X, Y) Z, \partial_s \rangle = 0, \\ \langle \text{Rm}(X, Y) Z, W \rangle &= \langle \text{Rm}^{\tilde{\pi}}(X, Y) Z, W \rangle - \frac{2a^2}{b^4} \langle IX, Y \rangle \langle IZ, W \rangle \\ &\quad + \frac{a^2}{b^4} \langle IY, Z \rangle \langle IX, W \rangle + \frac{a^2}{b^4} \langle IZ, X \rangle \langle IY, W \rangle. \end{aligned}$$

The first three equations imply (5.2) and (5.3). Meanwhile, observe that the last three equations imply

$$(5.7) \quad \text{Rm}(X, Y) Z = \text{Rm}^{\tilde{\pi}}(X, Y) Z - \frac{2a^2}{b^4} \langle IX, Y \rangle IZ + \frac{a^2}{b^4} (IX \wedge IY)(Z).$$

Since the base manifold of (5.1) is itself a warped product, we use the formulae for the curvature of a warped product and the Fubini–Study metric (e.g. [29, §IX.7]) to get

$$(5.8) \quad \begin{aligned} \text{Rm}^{\tilde{\pi}}(X, Y) Z &= \tilde{\pi}^*(\text{Rm}_{g_{\text{FS}}}(\pi_* X, \pi_* Y) \pi_* Z) + \frac{(b')^2}{b^2} (X \wedge Y)(Z) \\ &= \left(\frac{(b')^2 - 1}{b^2} \right) (X \wedge Y)(Z) + \frac{2}{b^2} \langle IX, Y \rangle IZ - \frac{1}{b^2} (IX \wedge IY)(Z). \end{aligned}$$

Combining (5.7) and (5.8), we arrive at (5.4). \square

Note that I turns $\mathcal{H}_p \cong T_p \mathbb{C}\mathbb{P}^{m-1}$ into a complex vector space, and together with the metric $g = \langle \cdot, \cdot \rangle$ it induces a Hermitian inner product $(X, Y) \mapsto \langle X, Y \rangle + i \langle IX, Y \rangle$. Denote by $SU(\mathcal{H}_p, I, g)$ the group of linear transformations on the complex vector space (\mathcal{H}_p, I) which preserve this Hermitian inner product, and let $\mathfrak{su}(\mathcal{H}_p, I, g)$ be its Lie algebra.

Corollary 5.10. *For all p in a dense open subset of M , there is a Lie subalgebra \mathfrak{k} of $\mathfrak{so}(T_p M)$ such that*

- (i) *Each map in \mathfrak{k} is a linear combination of compositions of maps of the form $\text{Rm}(X, Y)$ for some $X, Y \in \mathcal{H}_p$.*
- (ii) *Each map in \mathfrak{k} preserves the decomposition $T_p M = \mathcal{H}_p \oplus \mathcal{V}_p$.*
- (iii) *\mathfrak{k} acts on \mathcal{V}_p trivially, and acts on \mathcal{H}_p as the standard representation of $\mathfrak{su}(\mathcal{H}_p, I, g)$.*

Proof. Let $\{e_j\}_{j=1}^{m-1}$ be a unitary orthonormal basis for \mathcal{H}_p , so that $\{e_j, Ie_j\}_{j=1}^{m-1}$ is a g -orthonormal basis. Using Lemma 5.9, we compute for each $j \neq k$

$$\begin{aligned} A_{jk} &:= \text{Rm}(e_j, e_k) + \text{Rm}(Ie_j, Ie_k) = \left(\frac{(b')^2}{b^2} - \frac{2}{b^2} + \frac{a^2}{b^4} \right) (e_j \wedge e_k + Ie_j \wedge Ie_k), \\ B_{jk} &:= \text{Rm}(e_j, Ie_k) - \text{Rm}(Ie_j, e_k) = \left(\frac{(b')^2}{b^2} - \frac{2}{b^2} + \frac{a^2}{b^4} \right) (e_j \wedge Ie_k - Ie_j \wedge e_k), \\ D_{jk} &:= \text{Rm}(e_j, Ie_j) - \text{Rm}(e_k, Ie_k) = \left(\frac{(b')^2}{b^2} - \frac{2}{b^2} + \frac{a^2}{b^4} \right) (e_j \wedge Ie_j - e_k \wedge Ie_k). \end{aligned}$$

Suppose that $\frac{(b')^2}{b^2} - \frac{2}{b^2} + \frac{a^2}{b^4} \neq 0$ at p . Then A_{jk} , B_{jk} , and D_{jk} act trivially on \mathcal{V}_p , while they act on \mathcal{H}_p as a nonzero scalar multiple of

$$\begin{aligned} A_{jk} : \quad & e_j \mapsto -e_k, \quad e_k \mapsto e_j, \quad Ie_j \mapsto -Ie_k, \quad Ie_k \mapsto Ie_j, \\ B_{jk} : \quad & e_j \mapsto -Ie_k, \quad e_k \mapsto -Ie_j, \quad Ie_j \mapsto e_k, \quad Ie_k \mapsto e_j, \\ D_{jk} : \quad & e_j \mapsto -Ie_j, \quad e_k \mapsto Ie_k, \quad Ie_j \mapsto e_j, \quad Ie_k \mapsto -e_k. \end{aligned}$$

Thus, under the identification of complex vector spaces $(\mathcal{H}_p, I) \rightarrow (\mathbb{C}^{m-1}, i)$, these operators correspond to a generating set for $\mathfrak{su}(m-1)$ as follows:

$$A_{jk} \mapsto E_{jk} - E_{kj}, \quad B_{jk} \mapsto -i(E_{jk} + E_{kj}), \quad D_{jk} \mapsto i(E_{kk} - E_{jj}).$$

So $\{A_{jk}, B_{jk}, D_{jk}\}_{j,k=1,\dots,m-1; j \neq k}$ generates a Lie subalgebra \mathfrak{k} of $\mathfrak{so}(T_p M)$ with the desired properties.

Therefore, it remains to show that $\frac{(b')^2}{b^2} - \frac{2}{b^2} + \frac{a^2}{b^4} \neq 0$ on a dense open subset of M . This is equivalent to asking that $\varphi := b^2(b')^2 - 2b^2 + a^2 \neq 0$ on a dense open subset of $(0, \infty)$. Note that φ is analytic on $(0, \infty)$ by Proposition 5.7, and by the boundary conditions on a and b , $\varphi(s) \rightarrow -2b(0)^2 < 0$ as $s \rightarrow 0$. In particular, φ is nonzero for sufficiently small s , so it is nonzero on a dense open set. \square

Proof of Lemma 5.8. Since (M, g, J) is Kähler, we have $\nabla J = 0$. Thus, J commutes with all curvature operators $\text{Rm}(X, Y) : T_p M \rightarrow T_p M$. In particular, let $p \in M$ and $\mathfrak{k} \subset \mathfrak{so}(T_p M)$ be as in Corollary 5.10. Then J_p commutes with $\mathfrak{k} \subset \mathfrak{so}(T_p M)$. Using the properties of \mathfrak{k} from the corollary, it is easy to deduce that $J(\mathcal{V}_p) \subseteq \mathcal{V}_p$. Since J is invertible and orthogonal, this implies $J(\mathcal{V}_p) = \mathcal{V}_p$ and $J(\mathcal{H}_p) = \mathcal{H}_p$, i.e. J preserves the splitting $T_p M = \mathcal{H}_p \oplus \mathcal{V}_p$.

Recall that \mathfrak{k} acts as the standard representation of $\mathfrak{su}(\mathcal{H}_p, I, g) \cong \mathfrak{su}(m-1)$ on $\mathcal{H}_p \cong \mathbb{C}^{m-1}$. This is irreducible because $m \geq 3$, so its commutation with $J|_{\mathcal{H}_p}$ implies by Schur's lemma that $J|_{\mathcal{H}_p} = \pm i \cdot \text{Id}$ on \mathbb{C}^{m-1} , i.e. $J|_{\mathcal{H}_p} = \pm \tilde{\pi}^* J_{\text{FS}}$. Meanwhile, since $\dim_{\mathbb{C}} \mathcal{V}_p = 1$ and $J|_{\mathcal{V}_p}$ is orthogonal, the only two possibilities are $J|_{\mathcal{V}_p} = J_{\mathcal{V}}$ and $J|_{\mathcal{V}_p} = -J_{\mathcal{V}}$, where $J_{\mathcal{V}}$ is defined in the lemma. \square

We are now in a position to finish proving Theorem 1.4. By Proposition 5.7, it remains to analyze the case where (M, g, J) is diffeomorphic to $\mathcal{O}_{\mathbb{C}\mathbb{P}^{m-1}}(-m)$, and

$$(5.9) \quad g = ds^2 + g_{a(s), b(s)} = ds^2 + a(s)^2 \sigma \otimes \sigma + b(s)^2 \pi^* g_{\text{FS}}$$

on $M^{\text{prin}} = M \setminus \mathbb{C}\mathbb{P}^{m-1}$, where a and b are positive analytic functions on $(0, \infty)$. Using Lemma 5.8 and (5.9), we compute the Kähler form of (M, g, J) on a dense open subset of M^{prin} to be

$$(5.10) \quad \omega = \varepsilon_{\mathcal{V}} a(s) ds \wedge \sigma + \varepsilon_{\mathcal{H}} b(s)^2 \pi^* \omega_{\text{FS}},$$

where ω_{FS} is the Fubini–Study Kähler form, and $\varepsilon_{\mathcal{V}}, \varepsilon_{\mathcal{H}}$ are $\{1, -1\}$ -valued functions. Since (5.10) holds in a dense open set of M^{prin} , and ω is continuous, (5.10) actually holds everywhere on M^{prin} . Moreover, since M^{prin} is connected, the functions $\varepsilon_{\mathcal{V}}$ and $\varepsilon_{\mathcal{H}}$ must be constant. By imposing $d\omega = 0$ and using that $d\sigma = 2\pi^* \omega_{\text{FS}}$, we get

$$(-2\varepsilon_{\mathcal{V}} a + 2\varepsilon_{\mathcal{H}} b b') ds \wedge \pi^* \omega_{\text{FS}} = 0.$$

As $a, b > 0$, and b' is positive outside a compact set due to the ALE condition, this implies $\varepsilon_{\mathcal{V}} = \varepsilon_{\mathcal{H}}$ outside a compact set and hence everywhere. Thus,

$$\omega = \pm (a(s) ds \wedge \sigma + b(s)^2 \omega_{\text{FS}})$$

globally on M . Up to possible negation of the complex structure, this brings us to the exact situation of the Calabi ansatz; see §2.3, and in particular (2.5). From the discussion there, it follows that (M, g, J) is biholomorphically isometric to the Calabi space of dimension m , up to possibly replacing J with $-J$. This completes the proof of Theorem 1.4.

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