

# ON STEADY AND EXPANDING RICCI SOLITONS WITH ASYMPTOTIC SYMMETRIES

MICHAEL B. LAW

ABSTRACT. We establish a symmetry principle for asymptotically cylindrical steady gradient Ricci solitons (GRSs) and asymptotically conical expanding GRSs with homogeneous links. Using this, we show that the Bryant steady soliton is the unique asymptotically cylindrical steady GRS that has a round spherical link and satisfies a particular quantitative rigidity condition. A similar characterization is proved for Bryant’s expanding solitons. Finally, we establish a global symmetry result for GRSs which exhibit the aforementioned asymptotics with quotient-Berger sphere asymptotic links.

## CONTENTS

1. Introduction	1
2. Geometry of gradient Ricci solitons	5
3. Global symmetry principles	8
4. Uniqueness of rotationally symmetric GRSs	20
5. Global symmetries of GRSs with Berger-type asymptotic links	21
References	28

## 1. INTRODUCTION

Let  $(M^n, g)$  be a complete  $n$ -dimensional smooth Riemannian manifold, and let  $f : M \rightarrow \mathbb{R}$  be a smooth function. The triple  $(M^n, g, f)$  is called a *gradient Ricci soliton* (GRS) if

$$(1.1) \quad \text{Ric} = \nabla^2 f + \lambda g,$$

where  $\lambda$  determines the soliton type: *shrinking* ( $\lambda = \frac{1}{2}$ ), *steady* ( $\lambda = 0$ ), or *expanding* ( $\lambda = -\frac{1}{2}$ ). GRSs generate self-similar solutions of the Ricci flow and are fundamental to the analysis of singularities. In particular, shrinking and steady GRSs are special instances of *ancient solutions*, some of which are *singularity models*: blowup limits of Ricci flow on compact manifolds at finite-time singularities. On the other hand, steady and expanding GRSs are expected to play a key role in desingularization by geometric surgeries.

In dimension 3, all singularity models are *ancient  $\kappa$ -solutions*: non-flat,  $\kappa$ -noncollapsed ancient solutions with bounded and nonnegative sectional curvature. The classification of 3-dimensional ancient  $\kappa$ -solutions was completed by Brendle [6] and Brendle–Daskalopoulos–Sesum [7] in the noncompact and compact cases respectively. On the other hand, expanding GRSs play a less prominent role in dimension 3, and a full classification remains open. For more information on these developments, we refer to the books [16, 18] and references therein.

In dimensions  $\geq 4$ , ancient  $\kappa$ -solutions no longer model all finite-time singularities of Ricci flow: singularity models only have nonnegative *scalar* curvature in general. A full classification may therefore be unfeasible; indeed, even a classification of Einstein 4-manifolds

remains distant. Recent developments in mean curvature flow (e.g. [13, 19]) suggest that one should aim to only classify GRSs which are *stable*, since these are thought to govern singularity formation and resolution for *generic* solutions to Ricci flow.

To date, most classification results for GRSs and ancient solutions take place on smooth manifolds under assumptions on both asymptotic geometry and curvature; see for instance [5, 8, 12, 14, 20, 23, 37]. Motivated by the discussion above, our goal here is to investigate what remains true in the absence of curvature assumptions. Our results take place on *asymptotically cylindrical* steady GRSs and *asymptotically conical* expanding GRSs, defined below. For these GRSs, we will prove uniqueness and symmetry results without curvature assumptions, in exchange for an analytical condition on the weighted Lichnerowicz Laplacian (1.4). In accompanying work [30], we generalize our results to *Ricci-flat ALE spaces*, which themselves form a class of steady GRSs, but are also of independent interest.

**Definition 1.1.** *Let  $n \geq 3$ . Let  $\mathcal{S} = \mathbb{S}^{n-1}/\Gamma$  be a spherical space form of dimension  $n - 1$ . A steady GRS  $(M^n, g, f)$  is called asymptotically  $\mathcal{S}$ -cylindrical if the following holds. Let  $\Phi_t$  be the flow of  $\nabla f$ . Take an arbitrary sequence  $p_m$  of points going to infinity and consider the rescaled flows*

$$\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^* g,$$

with  $r_m R(p_m) = \frac{n-1}{2} + o(1)$ , where  $R(p_m)$  is the scalar curvature at  $p_m$ . We require that the flows  $(M, \hat{g}^{(m)}(t), p_m)$  converge as  $m \rightarrow \infty$  (without passing to a subsequence) in the Cheeger–Gromov sense to a family of shrinking cylinders  $(\mathcal{S} \times \mathbb{R}, \bar{g}(t))$ ,  $t \in (0, 1)$ , where

$$(1.2) \quad \bar{g}(t) = (n-2)(2-2t)\bar{g}_\infty + dz \otimes dz,$$

and  $\bar{g}_\infty$  is a round metric on  $\mathcal{S}$  with constant sectional curvature 1.

**Definition 1.2.** *Let  $n \geq 3$ , and let  $(\mathcal{S}, \bar{g}_\infty)$  be a closed Riemannian manifold of dimension  $n - 1$ . An expanding GRS  $(M^n, g, f)$  is called asymptotically  $(\mathcal{S}, \bar{g}_\infty)$ -conical if there is a compact set  $K \subset M$ , a radius  $R_0 > 0$ , a diffeomorphism  $\phi : (R_0, \infty) \times \mathcal{S} \rightarrow M \setminus K$ , and a decay rate  $\varepsilon > 0$  such that*

$$(1.3) \quad \begin{aligned} |\bar{\nabla}^k(\phi^* g - \bar{g})|_{\bar{g}} &\leq \mathcal{O}(r^{-\varepsilon-k}) \text{ as } r \rightarrow \infty \text{ for all } k \geq 0, \\ f(\phi(r, \theta)) &= \frac{r^2}{4} \text{ for all } (r, \theta) \in (R_0, \infty) \times \mathcal{S}, \end{aligned}$$

where  $\bar{g} = dr^2 + r^2 \bar{g}_\infty$  is the conical metric on  $(R_0, \infty) \times \mathcal{S}$ .

**Remark 1.3.**

- Brendle’s definition of asymptotic cylindricality [5] also requires that  $\frac{\Lambda_1}{r} \leq R \leq \frac{\Lambda_2}{r}$ , where  $\Lambda_1, \Lambda_2 > 0$  and  $r$  is the distance to a fixed point in  $M$ . It turns out that Definition 1.1 implies this; see [23, Lemma 6.5] and [17, §3].
- In Definition 1.1, we require  $\mathcal{S}$  to be a spherical space form and  $\bar{g}_\infty$  to have constant sectional curvature, but in Definition 1.2 we require neither.

For a symmetric 2-tensor  $h$  on a GRS  $(M^n, g, f)$  we define

$$(1.4) \quad Lh := \Delta h + \nabla_{\nabla f} h + 2 \text{Rm}(h),$$

where  $\text{Rm}(h)_{ik} = \text{Rm}_{ijkl} h^{jl}$  is the action of the curvature tensor. The operator  $L$  is formally self-adjoint in the space  $L^2(e^f)$  of symmetric 2-tensors satisfying  $\int_M |h|^2 e^f < \infty$ . Recall that  $L$  appears in the second variation of Perelman’s  $\nu$ -entropy, and  $Lh = 0$  is the linearization of the soliton equation (1.1) in a suitable gauge [11]. Accordingly, solutions of  $Lh = 0$  correspond to infinitesimal deformations of  $(M, g, f)$  that preserve the soliton condition to

first order. If the spectrum of  $L$  on  $L^2(e^f)$  is strictly positive, then  $(M, g, f)$  is said to be *strictly stable*. If  $\ker_{L^2(e^f)}(L) = \{0\}$ , then  $(M, g, f)$  is said to be *rigid*.

**1.1. A symmetry principle.** Our first result is the following symmetry principle. Given a Riemannian manifold  $(M, g)$ , denote by  $\mathcal{I}(M, g)$  its isometry group. Recall that  $(M, g)$  is *Riemannian homogeneous* if  $\mathcal{I}(M, g)$  acts transitively.

**Theorem 1.4.** *Let  $(M^n, g, f)$  be an asymptotically  $\mathcal{S}$ -cylindrical steady GRS or an asymptotically  $(\mathcal{S}, \bar{g}_\infty)$ -conical expanding GRS. Suppose  $(\mathcal{S}, \bar{g}_\infty)$  is Riemannian homogeneous. There exists  $d \geq 0$  such that if  $\dim \ker_{L^2(e^f)}(L) \leq d$ , then there is a compact Lie group  $G$  of dimension  $\geq \bar{d} - d$  which acts smoothly, effectively, isometrically, and with cohomogeneity one on  $(M, g)$ . The action preserves the level sets of  $f$ .*

The proof uses a generalization of Brendle's Killing field technique pioneered in [4, 5]. This technique forms the backbone of various uniqueness theorems in Ricci flow under positive curvature assumptions [6, 8, 12, 14, 37]. In these works, positive curvature is shown to imply rigidity, i.e.  $\ker_{L^2(e^f)}(L) = \{0\}$ , which then implies 'maximal' symmetry. Theorem 1.4 extends this, showing that without curvature assumptions, a dimension upper bound on  $\ker_{L^2(e^f)}(L)$  implies a lower bound on the number of global symmetries.

Next, we refine Theorem 1.4 for certain choices of asymptotic links  $(\mathcal{S}, \bar{g}_\infty)$ , in particular showing that all possibilities for  $G$  can be determined. This reduces  $(M, g)$  to a cohomogeneity one manifold with metric governed by an ODE system. Ultimately, this leads to classification and symmetry results for GRSs which we now describe.

**1.2. Uniqueness of rotationally symmetric GRSs.** If  $(\mathcal{S}, \bar{g}_\infty)$  is a round sphere, a refinement of Theorem 1.4 leads to the following result.

**Theorem 1.5.** *Let  $(M^n, g, f)$  be an asymptotically  $\mathbb{S}^{n-1}$ -cylindrical steady GRS of dimension  $n \geq 3$ . Suppose that*

$$(1.5) \quad \dim \ker_{L^2(e^f)}(L) \leq \begin{cases} n-2 & \text{if } n = 3 \text{ or } n \geq 5, \\ 1 & \text{if } n = 4. \end{cases}$$

*Then  $(M, g, f)$  is diffeomorphic to  $\mathbb{R}^n$  and rotationally symmetric, hence isometric to the Bryant steady soliton [9].*

*Similarly, suppose  $(M^n, g, f)$  is an asymptotically  $(\mathbb{S}^{n-1}, \alpha \bar{g}_\infty)$ -conical expanding GRS with  $n \geq 3$  and  $\alpha > 0$ , where  $\bar{g}_\infty$  is the round metric on  $\mathbb{S}^{n-1}$  with sectional curvature 1. If (1.5) holds, then  $(M, g, f)$  is diffeomorphic to  $\mathbb{R}^n$  and rotationally symmetric, hence isometric to the Bryant expanding GRS with cone angle  $\alpha$  [9].*

In the steady case, Brendle [4, 5] established the same uniqueness under a positive sectional curvature assumption instead of (1.5).<sup>1</sup> His proof establishes that  $\dim \ker_{L^2}(L) = \{0\}$ , so Theorem 1.5 generalizes his result. We note that Brendle's result has also been generalized to weaker curvature conditions [37] and on smooth orbifolds with positive sectional curvature [22].

In the expanding case, Chodosh [12] proved the same uniqueness under a positive sectional curvature assumption instead of (1.5), similarly to Brendle's theorem. Positive sectional curvature corresponds to  $\alpha < 1$ . However, Theorem 1.5 also applies to some nonpositively curved Bryant expanders ( $\alpha \geq 1$ ), as Deruelle [24] showed that there exists  $\varepsilon > 0$  such that strict stability (and hence  $\ker_{L^2(e^f)} = \{0\}$ ) holds when  $\alpha \in (0, 1 + \varepsilon)$ .

<sup>1</sup>In dimension 3, these hypotheses are met for any non-flat,  $\kappa$ -noncollapsed steady GRS. Brendle's result in this setting resolved a major conjecture left open in Perelman's first paper [34].

The special bound for  $n = 4$  in (1.5) arises from our proof strategy. In the course of the proof, we establish that  $G$  in Theorem 1.4 is a closed subgroup of  $SO(n)$  with dimension at least  $\frac{n(n-1)}{2} - \dim \ker_{L^2(ef)}(L)$ , but this does not imply  $G = SO(n)$  (and hence rotational symmetry) unless the bound (1.5) is met.

It remains unknown whether (1.5) can be dropped; this is true in mean curvature flow for the steady case, where the *bowl soliton* plays the role of the Bryant steady soliton [15, 27]. Theorem 1.5 says that if a counterexample exists, then it must admit a multi-parameter family of exponentially close steady GRSs at a linear level.

**1.3. Global symmetries of GRSs with Berger-type asymptotic links.** For  $m, k \geq 1$ , the principal  $U(1)$  Hopf fibration  $\pi : \mathbb{S}^{2m+1} \rightarrow \mathbb{C}\mathbb{P}^m$  induces a free action of the cyclic subgroup  $\mathbb{Z}_k \subset U(1)$  on  $\mathbb{S}^{2m+1}$ . Equip  $\mathbb{S}^{2m+1}$  with a Berger metric  $\tilde{g}_{a,b}$  for some  $a, b > 0$ , that is

$$\tilde{g}_{a,b} = a^2 \sigma \otimes \sigma + b^2 \pi^* g_{\text{FS}}$$

where  $\sigma$  is a 1-form dual to the Hopf fibers, and  $g_{\text{FS}}$  is the Fubini–Study metric on  $\mathbb{C}\mathbb{P}^m$  with holomorphic sectional curvature 4. Then the action of  $\mathbb{Z}_k$  is isometric on  $(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})$ , and the quotient is a lens space denoted by  $(L_{m,k}, g_{a,b})$ . When  $a = b = \kappa$ , the metrics  $\tilde{g}_{a,b}$  and  $g_{a,b}$  have constant sectional curvature  $\kappa^{-2}$ .

Consider metrics on  $(0, \infty) \times L_{m,k}$  of the form

$$g = ds^2 + g_{a(s),b(s)},$$

where  $a$  and  $b$  are smooth positive functions on  $(0, \infty)$ . A computation (e.g. [2, Appendix A]) shows that  $((0, \infty) \times L_{m,k}, g, f)$  defines a GRS (1.1) with  $f = f(s)$  whenever  $(f, a, b) : (0, \infty) \rightarrow \mathbb{R} \times (0, \infty)^2$  solves the system

$$(1.6) \quad f'' = -\frac{a''}{a} - 2m \frac{b''}{b} + \lambda,$$

$$(1.7) \quad a'' = 2m \left( \frac{a^3}{b^4} - \frac{a'b'}{b} \right) - a'f' + \lambda a,$$

$$(1.8) \quad b'' = \frac{2m+2}{b} - 2 \frac{a^2}{b^3} - \frac{a'b'}{a} - (2m-1) \frac{(b')^2}{b} - b'f' + \lambda b.$$

Moreover, if the following boundary conditions are satisfied:

$$(1.9) \quad f(0) = 0, \quad f'(0) = 0, \quad a(0) = 0, \quad a'(0) = k, \quad b(0) > 0, \quad b'(0) = 0,$$

then  $((0, \infty) \times L_{m,k}, g)$  is smoothly completed at  $s = 0$  by inserting a central  $\mathbb{C}\mathbb{P}^m$ . The resulting GRS  $(M, g, f)$  lives on a complete  $(2m+2)$ -dimensional Riemannian manifold diffeomorphic to the total space of the complex line bundle  $\mathcal{O}_{\mathbb{C}\mathbb{P}^m}(-k)$ .

For each  $m, k \geq 1$  and  $\alpha, \beta > 0$ , define the following classes of GRSs:

- $\mathcal{A}_{m,k}^s$  is the set of steady GRSs  $(M, g, f)$  defined by solutions  $(f, a, b) : (0, \infty) \rightarrow \mathbb{R} \times (0, \infty)^2$  of (1.6)–(1.9) with  $\lambda = 0$  such that  $\lim_{s \rightarrow \infty} \frac{a(s)}{b(s)} = 1$ .
- $\mathcal{A}_{m,k}^e(\alpha, \beta)$  is the set of expanding GRSs  $(M, g, f)$  defined by solutions  $(f, a, b) : (0, \infty) \rightarrow \mathbb{R} \times (0, \infty)^2$  of (1.6)–(1.9) with  $\lambda = -\frac{1}{2}$  such that  $\lim_{s \rightarrow \infty} \frac{a(s)}{s} = \alpha$  and  $\lim_{s \rightarrow \infty} \frac{b(s)}{s} = \beta$ .

Examples of GRSs in these classes include Appleton’s steady GRSs in  $\mathcal{A}_{m,k}^s$  for each  $m \geq 1$  and  $k \geq m+2$  [2], which are asymptotically  $(L_{m,k}, \bar{g}_\infty)$ -cylindrical. Appleton conjectured that there is exactly one soliton in each  $\mathcal{A}_{m,k}^s$  up to scaling. For expanders, Feldman, Ilmanen, and Knopf [26] proved the existence of a Kähler expanding GRS in  $\mathcal{A}_{m,k}^e(p, \sqrt{p})$

for each  $m \geq 1$ ,  $k \geq m + 2$ , and  $p > 0$ , unique up to scaling. These complement Cao's Kähler expanding GRSs on  $\mathbb{C}^n$  [10], for which a Brendle-type uniqueness theorem also holds [14]. More recently, Donovan [25] constructed families of 4-dimensional expanders in  $\bigcup_{\alpha, \beta > 0} \mathcal{A}_{1,k}^e(\alpha, \beta)$  for each  $k \geq 1$  which are generically non-Kähler and asymptotically conical, and contain the examples of [10, 26] as special cases.

In light of these constructions, we will establish the following symmetry reduction theorem, which is an analog of Theorem 1.5 for Berger-type asymptotic links. The special bounds for  $m = 1$  and  $m = 3$  are needed for similar reasons as above.

**Theorem 1.6.** *Let  $(M^{2m+2}, g, f)$  be an asymptotically  $(L_{m,k}, \bar{g}_\infty)$ -cylindrical steady GRS, where  $m \geq 1$  and  $k \geq 3$ . Suppose that*

$$(1.10) \quad \dim \ker_{L^2(\epsilon f)}(L) \leq \begin{cases} 2m - 1 & \text{if } m = 2 \text{ or } m \geq 4, \\ 2 & \text{if } m = 3, \\ 0 & \text{if } m = 1. \end{cases}$$

Then  $(M, g, f) \in \mathcal{A}_{m,k}^s$ .

Similarly, suppose  $(M^{2m+2}, g, f)$  is an asymptotically  $(L_{m,k}, g_{\alpha,\beta})$ -conical expanding GRS, where  $m \geq 1$ ,  $k \geq 3$ , and  $\alpha, \beta > 0$ . If (1.10) holds, then  $(M, g, f) \in \mathcal{A}_{m,k}^e(\alpha, \beta)$ .

A caveat of Theorem 1.6 is that we do not know whether the existing examples satisfy (1.10), though a relevant analysis has been carried out for the much-related *FIK shrinker* in dimension 4 [33]. We also refer to our other work [30] for an analog of Theorem 1.6 in the setting of Ricci-flat Kähler ALE spaces, where it implies a genuine uniqueness result for certain examples.

**1.4. Structure of the paper.** In §2, we collect known facts about asymptotically cylindrical steady GRSs and asymptotically conical expanding GRSs. In §3, we prove Theorem 1.4 by generalizing Brendle's Killing field technique. In §4, we apply the results from the previous sections to prove Theorem 1.5. In §5, we proceed along similar lines to prove Theorem 1.6.

**Acknowledgements.** We thank William Minicozzi and Tristan Ozuch for their insights and continual support, as well as Shrey Aryan, Pak-Yeung Chan, Yonghwan Kim, and Yongjia Zhang for inspiring discussions. We are grateful to Bennett Chow and Yuxing Deng for noticing an error in a previous version of this paper. This work was partially supported by a Croucher Scholarship and by a Simons Dissertation Fellowship in Mathematics.

## 2. GEOMETRY OF GRADIENT RICCI SOLITONS

This section collects known geometric properties of the spaces we study.

**2.1. Asymptotically cylindrical steady GRSs.** In this subsection, we assume  $(M^n, g, f)$  is an asymptotically  $\mathcal{S}$ -cylindrical steady GRS of dimension  $n \geq 3$  (see Definition 1.1). Recall that the steady GRS equation is

$$(2.1) \quad \text{Ric} = \nabla^2 f.$$

Denote by  $R$  the scalar curvature of  $(M, g)$ . Standard identities for steady GRSs give  $R \geq 0$  and  $R + |\nabla f|^2 \equiv c \geq 0$ , and if equality holds in either then  $\text{Ric} = 0$ . Clearly, asymptotically  $\mathcal{S}$ -cylindrical steady GRSs are not Ricci flat, so

$$R > 0$$

and by scaling the metric we may arrange that  $c = 1$ ; thus

$$(2.2) \quad R + |\nabla f|^2 = 1.$$

Our convention for the Laplacian on any tensor is  $\Delta = -\nabla^* \nabla$ . The drift Laplacian  $\Delta_f$  on tensors is defined by

$$\Delta_f = \Delta + \nabla_{\nabla f}.$$

This is self-adjoint with respect to the weighted volume form  $e^f \text{dvol}_g$ . Under the normalization (2.2), one has the soliton identity

$$\Delta_f f = 1.$$

Let  $r : M \rightarrow \mathbb{R}$  be the distance from a fixed point with respect to  $g$ .

**Lemma 2.1** ([17, §2], also [3, Lemma 3.14]). *We have  $\frac{f(x)}{r(x)} \rightarrow 1$  as  $r(x) \rightarrow \infty$ .*

In particular,  $f$  is proper and bounded from below. As (2.1) and (2.2) are invariant under translating  $f$ , we will henceforth assume  $\min_M f = 1$ .

**Lemma 2.2.** *For sufficiently large  $\rho > 0$ , we have  $\nabla f \neq 0$  on  $\{f \geq \rho\}$ . Hence,  $\{f = \rho\}$  is a smooth hypersurface in  $M$ .*

*Proof.* By [23, Lemma 6.5], asymptotic  $\mathcal{S}$ -cylindricity implies that  $R = o(1)$  as  $r \rightarrow \infty$ . Then the claim follows from (2.2) and Lemma 2.1.  $\square$

Next, we state the roundness estimates on the level sets of  $f$  which were first established by Brendle [5] and subsequently improved by Zhao–Zhu [37].

**Proposition 2.3** ([37, Propositions 2.5 and 2.7]). *Let  $\{e_1, \dots, e_n\}$  be an orthonormal frame such that  $e_n = \frac{\nabla f}{|\nabla f|}$ . There exists  $\delta > 0$  such that on  $M$ , we have*

$$fR = \frac{n-1}{2} + \mathcal{O}(r^{-\frac{1}{2}-\delta})$$

and

$$\begin{aligned} f \text{Rm}_{ijkl} &= \frac{1}{2(n-2)} (g_{ik} - \partial_i f \partial_k f) (g_{jl} - \partial_j f \partial_l f) \\ &\quad - \frac{1}{2(n-2)} (g_{il} - \partial_i f \partial_l f) (g_{jk} - \partial_j f \partial_k f) + \mathcal{O}(r^{-\frac{1}{2}-\delta}). \end{aligned}$$

For the full proof of this result, see Propositions 2.1–2.5 in [5] followed by Section 2.1 of [37]. Once  $f$  and  $r$  are known to be comparable (as in Lemma 2.1), these proofs only use asymptotic cylindricity and do not require any curvature hypotheses. They carry over verbatim to asymptotically  $\mathcal{S}$ -cylindrical steady GRSs.

**Corollary 2.4.** *We have  $|\nabla^k \text{Rm}| \leq \mathcal{O}(r^{-1-\frac{k}{2}})$  for each  $k \geq 0$ . Also,  $|\nabla f| \leq \mathcal{O}(1)$  and  $|\nabla^k f| \leq \mathcal{O}(r^{-\frac{k}{2}})$  for each  $k \geq 2$ .*

*Proof.* The curvature bounds follow from Lemma 2.1, Proposition 2.3, and Shi's estimates. Combining this with (2.2) and the equation  $\text{Ric} = \nabla^2 f$ , the bounds on  $f$  follow.  $\square$

**Theorem 2.5.**  *$M$  has exactly one end.*

*Proof.* This follows from [32, Corollary 1.1] since  $(M, g)$  clearly does not split off a line.  $\square$

**Corollary 2.6.** *The hypersurfaces  $\{f = \rho\}$  are diffeomorphic to  $\mathcal{S}$  for all large  $\rho$ .*

*Proof.* By Lemma 2.1 and Lemma 2.2, the level sets  $\{f = \rho\}$  are smooth hypersurfaces bounding the end(s) of  $M$ . Then by Theorem 2.5,  $\{f = \rho\}$  is connected for all large  $\rho$ .

Take a sequence of points  $p_m$  going to infinity, and let  $r_m = f(p_m)$ ,  $\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^* g$ . By [23, Lemma 6.5], the Ricci curvature of the level sets of  $f$  decays linearly in  $r$ . Then by the Bonnet–Myers theorem, the  $g$ -diameter of  $\{f = r_m\}$  is bounded by  $C\sqrt{r_m}$ , so the  $\hat{g}^{(m)}(t)$ -diameter of  $\{f = r_m\}$  is uniformly bounded over all  $m$  and over all  $t$  in a compact interval within  $(0, 1)$ .

By asymptotic  $\mathcal{S}$ -cylindricity, the sequence  $(M, \hat{g}^{(m)}(t), p_m)$  converges in the Cheeger–Gromov sense to a shrinking  $\mathcal{S}$ -cylinder (1.2) for  $t \in (0, 1)$ . Moreover, the vector fields  $\sqrt{r_m} \nabla f$  converge to the axial vector field  $\frac{\partial}{\partial z}$  on  $\mathcal{S} \times \mathbb{R}$ . Using this, and the assertions on the  $\hat{g}^{(m)}(t)$ -diameter of  $\{f = r_m\}$  from the last paragraph, it is easy to see that  $\{f = r_m\}$  is diffeomorphic to  $\mathcal{S}$  for all large  $m$ .  $\square$

By the above results,  $f$  is a coordinate on the end of  $M$  with level sets diffeomorphic to  $\mathcal{S}$ . Let  $F_\rho : \mathcal{S} \rightarrow \{f = \rho\} \subset M$  be a one-parameter family of diffeomorphisms such that  $\frac{\partial}{\partial \rho} F_\rho = \frac{\nabla f}{|\nabla f|^2}$ . Define Riemannian metrics  $g_\rho$  on  $\mathcal{S}$  by  $g_\rho = F_\rho^* g$ . Then the end of  $(M, g)$  is isometric to

$$(2.3) \quad \left( I \times \mathcal{S}, \frac{df^2}{|\nabla f|^2} + g_f \right),$$

where  $I$  is an interval of the form  $[\rho_0, \infty)$ . Following [37, §2.2], we have the following convergence and closeness bounds:

**Lemma 2.7.** *There exists  $\delta > 0$  such that for each  $k \geq 0$ , we have*

$$\left\| \frac{1}{2(n-2)\rho} g_\rho - \bar{g}_\infty \right\|_{C^k(\mathcal{S}, \bar{g}_\infty)} \leq \mathcal{O}(\rho^{-\frac{1}{2} - \frac{1}{2}\delta})$$

as  $\rho \rightarrow \infty$ , where  $\bar{g}_\infty$  is a metric on  $\mathcal{S}$  with constant sectional curvature 1.

Thus,  $(M, g)$  opens up like a paraboloid, and has volume growth of order  $r^{\frac{n+1}{2}}$ .

In what follows, given an asymptotically  $\mathcal{S}$ -cylindrical steady GRS  $(M^n, g, f)$ , we will identify level sets  $\{f = \rho\}$  with  $(\mathcal{S}, g_\rho)$  and the end of  $(M, g)$  with (2.3). The metric  $\bar{g}_\infty$  is understood to come from Lemma 2.7.

**2.2. Asymptotically conical expanding GRSs.** In this subsection, we assume  $(M^n, g, f)$  is an asymptotically  $(\mathcal{S}, \bar{g}_\infty)$ -conical expanding GRS (see Definition 1.2). Recall that the expanding GRS equation is

$$\text{Ric} = \nabla^2 f - \frac{1}{2}g.$$

Standard identities for expanding GRSs give

$$(2.4) \quad R + \frac{n}{2} \geq 0,$$

$$(2.5) \quad R + |\nabla f|^2 = f + \mu,$$

$$(2.6) \quad \Delta_f f = f + \frac{n}{2} + \mu,$$

where  $\mu \in \mathbb{R}$  is a constant. We have the following analogs of the results from the last subsection. The first one follows directly from the definition. Recall that  $r : M \rightarrow \mathbb{R}$  denotes the distance from a fixed point with respect to  $g$ .

**Lemma 2.8.** *We have  $f(x) \sim \frac{r(x)^2}{4}$  as  $r(x) \rightarrow \infty$ . For sufficiently large  $\rho > 0$ , we have  $\nabla f \neq 0$  on  $\{f \geq \rho\}$ . Hence,  $\{f = \rho\}$  is a smooth hypersurface diffeomorphic to  $\mathcal{S}$ .*

**Lemma 2.9.** *We have  $|\nabla^k \text{Rm}| \leq \mathcal{O}(r^{-\varepsilon-k})$  and  $|\nabla^k f| \leq \mathcal{O}(r^{2-k})$  for each  $k \geq 0$ .*

*Proof.* The bound on  $|\nabla^k \text{Rm}|$  follows from (1.3). Using (2.5) and the fact that  $f \sim \frac{r^2}{4}$ , we have  $|\nabla f| = \mathcal{O}(r)$ . Also  $|\nabla^2 f| = |\text{Ric} + \frac{1}{2}g| \leq \mathcal{O}(1)$ . Finally, for each  $k \geq 3$ , we have  $|\nabla^k f| = |\nabla^{k-2} \text{Ric}| \leq \mathcal{O}(r^{-\varepsilon+2-k}) \leq \mathcal{O}(r^{2-k})$ .  $\square$

**Lemma 2.10.** *For each sufficiently large  $\rho > 0$ , identify  $\{f = \rho\}$  with the Riemannian manifold  $(\mathcal{S}, g_\rho)$ . Then for each  $k \geq 0$ , we have*

$$\left\| \frac{1}{4\rho} g_\rho - \bar{g}_\infty \right\|_{C^k(\mathcal{S}, \bar{g}_\infty)} \leq \mathcal{O}(\rho^{-\varepsilon}).$$

### 3. GLOBAL SYMMETRY PRINCIPLES

The goal of this section is to prove Theorem 1.4.

**3.1. Killing fields on asymptotically cylindrical steady GRSs.** In this subsection, we fix an asymptotically  $\mathcal{S}$ -cylindrical steady GRS  $(M^n, g, f)$ . We first pull back Killing fields from  $(\mathcal{S}, \bar{g}_\infty)$  to obtain approximate Killing fields on  $(M, g)$ . Given a Killing field  $\bar{U}$  on  $(\mathcal{S}, \bar{g}_\infty)$ , we will say that a vector field  $U$  on  $M$  *extends*  $\bar{U}$  if its restriction to any sufficiently far level set  $\{f = \rho\} \cong \mathcal{S}$  satisfies

$$U|_{\{f=\rho\}} = \bar{U}.$$

**Proposition 3.1.** *Let  $\bar{U}$  be a Killing field on  $(\mathcal{S}, \bar{g}_\infty)$ , and let  $U$  be a smooth vector field on  $M$  which extends  $\bar{U}$ . Then there is a smooth vector field  $W$  on  $M$  such that*

$$\begin{aligned} \Delta_f W &= 0, \\ |\nabla^k(W - U)| &\leq \mathcal{O}(r^{-\frac{k}{2}}) \text{ for each } k \geq 0, \\ |\nabla^k W| &\leq \mathcal{O}(r^{\frac{1}{2} - \frac{k}{2}}) \text{ for each } k \geq 0, \\ |\mathcal{L}_W g| &\leq \mathcal{O}(r^{-\frac{1}{2}}). \end{aligned}$$

*Proof.* From [37, Proposition 3.1] and its proof, we have the following estimates for some  $\delta > 0$ :

$$(3.1) \quad |\mathcal{L}_U g| \leq \mathcal{O}(r^{-\frac{1}{2} - \frac{1}{2}\delta}),$$

$$(3.2) \quad |\nabla^k \mathcal{L}_U g| \leq \mathcal{O}(r^{-\frac{k}{2}}) \text{ for each } k \geq 1,$$

$$(3.3) \quad |\nabla^k U| \leq \mathcal{O}(r^{\frac{1}{2} - \frac{k}{2}}) \text{ for each } k \geq 0,$$

$$(3.4) \quad |\Delta_f U| \leq \mathcal{O}(r^{-1 - \frac{1}{4}\delta}).$$

Combining (3.2) with the relation

$$(3.5) \quad \text{div}(\mathcal{L}_U g) - \frac{1}{2} \nabla(\text{tr} \mathcal{L}_U g) = \Delta U + \text{Ric}(U)$$

yields

$$(3.6) \quad |\nabla^k(\Delta U + \text{Ric}(U))| \leq \mathcal{O}(r^{-\frac{1}{2} - \frac{k}{2}}) \text{ for each } k \geq 0.$$

Since  $[U, \frac{\nabla f}{|\nabla f|^2}] = 0$  outside a compact set, we have

$$\nabla_{\nabla f} U - \text{Ric}(U) = -[U, \nabla f] = -U(|\nabla f|^2) \frac{\nabla f}{|\nabla f|^2} = \langle U, \nabla R \rangle \frac{\nabla f}{|\nabla f|^2}$$

outside a compact set. By (3.3) and Corollary 2.4, this implies

$$(3.7) \quad |\nabla^k(\nabla_{\nabla f} U - \text{Ric}(U))| \leq \mathcal{O}(r^{-1-\frac{k}{2}}) \quad \text{for each } k \geq 0.$$

Adding (3.6) and (3.7) together, we get

$$|\nabla^k \Delta_f U| \leq \mathcal{O}(r^{-\frac{1}{2}-\frac{k}{2}}) \quad \text{for each } k \geq 0.$$

Using this and (3.4) with standard interpolation inequalities gives

$$|\nabla^k \Delta_f U| \leq \mathcal{O}(r^{-1-\frac{k}{2}}) \quad \text{for each } k \geq 0.$$

Using the estimate (3.4), it is proved in Lemma 3.2 and Theorem 3.3 in [37] that there exists a solution  $V$  of

$$(3.8) \quad \Delta_f V = -\Delta_f U$$

such that  $|V| \leq \mathcal{O}(1)$ . Set  $W := U + V$ . Standard interior parabolic Schauder estimates applied to (3.8) yields  $|\nabla^k V| \leq \mathcal{O}(r^{-\frac{k}{2}})$ . Combining this with (3.1) and (3.3) gives the claimed estimates.  $\square$

The main result of this subsection is Proposition 3.8, which upgrades some of the approximate Killing fields  $W$  from Proposition 3.1 to exact Killing fields. The key is to study a PDE satisfied by the Lie derivative of  $W$ . We define the operator  $L$  on symmetric 2-tensors  $h$  by

$$Lh = \Delta_f h + 2\text{Rm}(h),$$

where  $\text{Rm}(h)_{ik} = \text{Rm}_{ijkl} h^{jl}$ . As  $\text{Ric} = \nabla^2 f$ , one easily checks that

$$Lh = \Delta_L h + \mathcal{L}_{\nabla f} h,$$

where  $\Delta_L$  is the Lichnerowicz Laplacian defined by

$$(3.9) \quad \Delta_L h_{ik} = \Delta h_{ik} + 2\text{Rm}_{ijkl} h^{jl} - \text{Ric}_i^l h_{kl} - \text{Ric}_k^l h_{il}.$$

Brendle made the following crucial observations:

**Theorem 3.2** ([4, §4]). *Let  $W$  be a vector field satisfying  $\Delta_f W = 0$ . Then  $h = \mathcal{L}_W g$  satisfies  $Lh = 0$ . In particular, taking  $W = \nabla f$  gives  $L\text{Ric} = 0$ .*

We will prove strong decay estimates for solutions of  $Lh = 0$ , starting with a computation:

**Lemma 3.3.** *For all sufficiently large  $\ell > 0$ , there exists a compact set  $K_\ell \subset M$  such that for all symmetric 2-tensors  $h$  on  $M$  satisfying  $Lh = 0$ , we have*

$$\Delta_f(f^\ell |h|^2) \geq 0 \quad \text{on } M \setminus K_\ell.$$

*Proof.* Since  $Lh = 0$ , we have

$$\Delta_f |h|^2 = 2\langle \Delta_f h, h \rangle + 2|\nabla h|^2 = -4\langle \text{Rm}(h), h \rangle + 2|\nabla h|^2.$$

Let  $\ell > 0$ . Using this and the soliton identity  $\Delta_f f = 1$ , we get

$$\begin{aligned} \Delta_f(f^\ell |h|^2) &= (\Delta_f f^\ell) |h|^2 + f^\ell \Delta_f |h|^2 + 2\langle \nabla f^\ell, \nabla |h|^2 \rangle \\ &= (\ell f^{\ell-1} + \ell(\ell-1)f^{\ell-2} |\nabla f|^2) |h|^2 \\ &\quad + f^\ell (2|\nabla h|^2 - 4\langle \text{Rm}(h), h \rangle) + 2\ell f^{\ell-1} \langle \nabla f, \nabla |h|^2 \rangle \end{aligned}$$

$$(3.10) \quad \begin{aligned} &\geq \ell f^{\ell-1} |h|^2 + \ell(\ell-1) f^{\ell-2} |\nabla f|^2 |h|^2 + 2f^\ell |\nabla h|^2 \\ &\quad - 4f^\ell |\text{Rm}| |h|^2 - 4\ell f^{\ell-1} |\nabla f| |h| |\nabla h|. \end{aligned}$$

By Proposition 2.3, we have

$$|\text{Rm}| \leq C f^{-1}.$$

Moreover, by the absorbing inequality and Kato's inequality, we have

$$2|\nabla f| |h| |\nabla h| \leq \frac{f}{\ell} |\nabla h|^2 + \frac{\ell}{f} |\nabla f|^2 |h|^2.$$

Using these facts in (3.10) gives

$$\begin{aligned} \Delta_f (f^\ell |h|^2) &\geq (\ell - C) f^{\ell-1} |h|^2 - (\ell + \ell^2) f^{\ell-2} |\nabla f|^2 |h|^2 \\ &= f^{\ell-2} |h|^2 [(\ell - C) f - (\ell + \ell^2) |\nabla f|^2]. \end{aligned}$$

Since  $|\nabla f|^2 \leq 1$ , and  $f \rightarrow +\infty$  at infinity, it follows that if  $\ell > C$ , the right-hand side is nonnegative outside a compact set of  $M$ .  $\square$

The next two results are due to Brendle and Zhao–Zhu, respectively.

**Proposition 3.4** ([5, Lemma 4.1]). *Consider the shrinking cylinder  $(\mathcal{S} \times \mathbb{R}, \bar{g}(t))$ ,  $t \in (0, 1)$ , where  $\bar{g}(t)$  is given by (1.2). Let  $\bar{h}(t)$ ,  $t \in (0, 1)$ , be a one-parameter family of symmetric 2-tensors which solve the parabolic equation*

$$\frac{\partial}{\partial t} \bar{h}(t) = \Delta_{L, \bar{g}(t)} \bar{h}(t),$$

where  $\Delta_{L, \bar{g}(t)}$  is the Lichnerowicz Laplacian (see (3.9)). Moreover, suppose  $\bar{h}(t)$  is invariant under translations along the axis of the cylinder, and

$$|\bar{h}(t)|_{\bar{g}(t)} \leq (1-t)^{-\ell}$$

for all  $t \in (0, \frac{1}{2}]$  and some  $\ell > 0$ . Then there exists  $N > 0$  depending on  $\ell$  such that

$$\inf_{\lambda \in \mathbb{R}} \sup_{\mathcal{S} \times \mathbb{R}} |\bar{h}(t) - \lambda \text{Ric}_{\bar{g}(t)}|_{\bar{g}(t)} \leq N(1-t)^{\frac{1}{2(n-2)} - \frac{1}{2}}$$

for all  $t \in [\frac{1}{2}, 1)$ .

*Proof.* The proof of [5, Lemma 4.1] applies verbatim after replacing every occurrence of  $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$  there with  $(\mathcal{S}, \bar{g}_\infty)$ . (The proof uses spectral properties of the Laplacian which still hold on  $(\mathcal{S}, \bar{g}_\infty)$ .)  $\square$

**Proposition 3.5** ([37, Theorem 4.3]). *If  $Lh = 0$  on  $M$  and  $|h| \leq \mathcal{O}(r^{-\frac{1}{2}})$ , then there exists  $\lambda \in \mathbb{R}$  such that  $|h - \lambda \text{Ric}| \leq \mathcal{O}(r^{-\ell})$  for every  $\ell > 0$ .*

*Proof sketch.* Due to differences in organization, we will outline the proof, omitting steps which can essentially be copied down from the original source.

Let  $h$  satisfy  $Lh = 0$  and  $|h| \leq \mathcal{O}(r^{-\frac{1}{2}})$ . Consider the function

$$A(\rho) = \inf_{\lambda \in \mathbb{R}} \sup_{\{f=\rho\}} |h - \lambda \text{Ric}|.$$

Let  $r_m \rightarrow \infty$  be a sequence of numbers, and denote by  $\lambda_m$  the minimizing  $\lambda$ 's associated with  $A(r_m)$ .

By Theorem 3.2, we have  $L(h - \lambda_m \text{Ric}) = 0$  for each  $m$ . Then by Lemma 3.3 and the maximum principle, for each large  $\ell > 0$  there exists  $\rho_\ell > 0$  such that for all  $m$ ,

$$\sup_{\{\rho_\ell \leq f \leq r_m\}} f^\ell |h - \lambda_m \text{Ric}|^2 \leq \max \left\{ \rho_\ell^\ell \sup_{\{f=\rho_\ell\}} |h - \lambda_m \text{Ric}|^2, r_m^\ell \sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}|^2 \right\}.$$

Taking square roots and redefining  $\rho_\ell$  appropriately, this becomes

$$(3.11) \quad \sup_{\{\rho_\ell \leq f \leq r_m\}} f^\ell |h - \lambda_m \text{Ric}| \leq \max \left\{ \rho_\ell^\ell \sup_{\{f=\rho_\ell\}} |h - \lambda_m \text{Ric}|, r_m^\ell \sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}| \right\}.$$

We divide into two cases:

**Case 1:** there exists  $\ell > 0$  and a subsequence of  $r_m$  (still denoted  $r_m$ ) such that

$$\rho_\ell^\ell \sup_{\{f=\rho_\ell\}} |h - \lambda_m \text{Ric}| \leq r_m^\ell \sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}|.$$

This is Case 1 in the proof of [37, Theorem 4.3]. Following the argument there (which uses Proposition 3.4), we see that the  $\lambda_m$ 's are all equal to a constant  $\lambda$ , and

$$h = \lambda \text{Ric} \quad \text{on } \{f \geq \rho_\ell\}.$$

This clearly implies  $|h - \lambda \text{Ric}| \leq \mathcal{O}(r^{-\ell})$  for every  $\ell > 0$ .

**Case 2:** for all  $\ell > 0$ , there exists  $R_\ell > 0$  such that for all  $r_m > R_\ell$ ,

$$\rho_\ell^\ell \sup_{\{f=\rho_\ell\}} |h - \lambda_m \text{Ric}| > r_m^\ell \sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}|.$$

Then by (3.11), for each  $\ell > 0$ , and for all  $m$  such that  $r_m > R_\ell$ , one has

$$(3.12) \quad \sup_{\{\rho_\ell \leq f \leq r_m\}} f^\ell |h - \lambda_m \text{Ric}| \leq \rho_\ell^\ell \sup_{\{f=\rho_\ell\}} |h - \lambda_m \text{Ric}|.$$

Following the arguments in Case 2 in the proof of [37, Theorem 4.3] (in particular Claim 4.4 there), we conclude that the sequence  $\lambda_m$  is uniformly bounded, so a subsequence of the  $\lambda_m$  converges to some  $\lambda \in \mathbb{R}$ . Fixing an arbitrary  $\ell > 0$  in (3.12) and taking  $m \rightarrow \infty$ , we get

$$\sup_{\{f \geq \rho_\ell\}} f^\ell |h - \lambda \text{Ric}| \leq \rho_\ell^\ell \sup_{\{f=\rho_\ell\}} |h - \lambda \text{Ric}|.$$

Hence, for all  $\rho \geq \rho_\ell$ ,

$$\sup_{\{f=\rho\}} |h - \lambda \text{Ric}| \leq \frac{\rho_\ell^\ell}{\rho^\ell} \sup_{\{f=\rho_\ell\}} |h - \lambda \text{Ric}| \leq C_\ell \rho^{-\ell}$$

where  $C_\ell > 0$  depends on  $\ell$ . So  $|h - \lambda \text{Ric}| \leq \mathcal{O}(r^{-\ell})$  for each  $\ell > 0$ .  $\square$

Finally, we record a simple application of soliton identities and the maximum principle.

**Lemma 3.6.** *Suppose  $u$  is a function such that  $\Delta_f u \geq 0$  outside a compact set and  $u \rightarrow 0$  at infinity. Then  $u \leq Ce^{-f}$  on  $M$  for some  $C > 0$ .*

*Proof.* Recall that  $R \geq 0$  on any steady GRS. By this and (2.2),

$$\Delta_f e^{-f} = -e^{-f} \Delta_f f + e^{-f} |\nabla f|^2 = e^{-f} (-1 + |\nabla f|^2) = -Re^{-f} \leq 0.$$

Choose  $\rho > 0$  large so that  $\Delta_f u \geq 0$  on  $\{f > \rho\}$ . Choose  $C > 0$  large so that  $u - Ce^{-f} < 0$  on  $\{f = \rho\}$ . We have

$$\Delta_f (u - Ce^{-f}) \geq 0 \quad \text{on } \{f > \rho\},$$

and  $u - Ce^{-f} \rightarrow 0$  at infinity. It follows by the maximum principle that  $u - Ce^{-f} \leq 0$  on  $\{f > \rho\}$ . Suitably increasing  $C$ , the lemma follows.  $\square$

We can now prove exponential decay of solutions to  $Lh = 0$  up to a multiple of the Ricci tensor.

**Lemma 3.7.** *If  $Lh = 0$  on  $M$  and  $|h| \leq \mathcal{O}(r^{-\frac{1}{2}})$ , then  $h - \lambda \text{Ric} \in L^2(e^f)$  for some  $\lambda \in \mathbb{R}$ .*

*Proof.* Let  $h$  be as in the lemma. By Proposition 3.5, we can find  $\lambda \in \mathbb{R}$  such that  $\tilde{h} = h - \lambda \text{Ric}$  satisfies  $f^\ell |\tilde{h}|^2 \rightarrow 0$  at infinity for all  $\ell > 0$ . Moreover,  $L\tilde{h} = 0$  by Theorem 3.2. By Lemma 3.3, it holds for each sufficiently large  $\ell > 0$  that

$$\Delta_f(f^\ell |\tilde{h}|^2) \geq 0 \quad \text{outside a compact set.}$$

Then Lemma 3.6 gives  $f^\ell |\tilde{h}|^2 \leq \mathcal{O}(e^{-f})$  outside a compact set. Since  $f \sim r$  and  $(M, g)$  has volume growth of order  $r^{\frac{n+1}{2}}$ , this implies  $\tilde{h} \in L^2(e^f)$ .  $\square$

Now we find exact Killing fields on  $(M, g)$ , following [4, Theorem 7.1]; the number of these will depend on a dimension bound for  $\ker_{L^2(e^f)}(L)$ . We let  $\bar{d}$  denote the dimension of the space of Killing fields on the link  $(\mathcal{S}, \bar{g}_\infty)$ , equivalently the dimension of its isometry group.

**Proposition 3.8.** *Suppose  $\dim \ker_{L^2(e^f)}(L) \leq d$  for some nonnegative integer  $d$ . Then there exists a  $(\bar{d} - d)$ -dimensional vector space  $\mathcal{V}$  of Killing fields on  $(M, g)$  such that for each  $W \in \mathcal{V}$ ,*

- (a)  $[W, \nabla f] = 0$ .
- (b)  $\langle W, \nabla f \rangle = 0$ , i.e.  $W$  is tangent to the level sets of  $f$ .
- (c)  $W$  is a Killing field for  $(M, g)$  and for each sufficiently far level set of  $f$ .
- (d)  $|\nabla^k W| \leq \mathcal{O}(r^{\frac{1}{2} - \frac{k}{2}})$  for each  $k \geq 0$ .
- (e) There is a unique Killing field  $\bar{U}$  on  $(\mathcal{S}, \bar{g}_\infty)$  such that if  $U$  is a smooth vector field on  $M$  which extends  $\bar{U}$ , then  $|\nabla^k(W - U)| \leq \mathcal{O}(r^{-\frac{k}{2}})$  for each  $k \geq 0$ . The map  $\mathcal{V} \rightarrow \mathfrak{I}(\mathcal{S}, \bar{g}_\infty)$ ,  $W \mapsto \bar{U}$  is linear and injective.

*Proof.* Let  $\{\bar{U}_i\}_{i=1}^{\bar{d}}$  be a basis for the space of Killing fields  $\mathfrak{I}(\mathcal{S}, \bar{g}_\infty)$ , and let  $\{U_i\}_{i=1}^{\bar{d}}$  be smooth vector fields on  $M$  which extend  $\bar{U}_i$ . Proposition 3.1 gives vector fields  $\{\tilde{W}_i\}_{i=1}^{\bar{d}}$  such that

$$\begin{aligned} \Delta_f \tilde{W}_i &= 0, \\ (3.13) \quad |\nabla^k(\tilde{W}_i - U_i)| &\leq \mathcal{O}(r^{-\frac{k}{2}}) \text{ for each } k \geq 0, \\ |\nabla^k \tilde{W}_i| &\leq \mathcal{O}(r^{\frac{1}{2} - \frac{k}{2}}) \text{ for each } k \geq 0, \\ |\mathcal{L}_{\tilde{W}_i} g| &\leq \mathcal{O}(r^{-\frac{1}{2}}). \end{aligned}$$

By Theorem 3.2, we have  $L(\mathcal{L}_{\tilde{W}_i} g) = 0$  and  $L(\mathcal{L}_{\nabla f} g) = 0$ . Then Lemma 3.7 gives  $\lambda_i \in \mathbb{R}$  such that  $W_i := \tilde{W}_i - \lambda_i \nabla f$  satisfy  $\mathcal{L}_{W_i} g \in \ker_{L^2(e^f)}(L)$ . By the weighted Bochner formula and Corollary 2.4, we see that

$$\begin{aligned} \Delta_f W_i &= 0, \\ (3.14) \quad |\nabla^k(W_i - U_i)| &\leq \mathcal{O}(r^{-\frac{k}{2}}) \text{ for each } k \geq 0, \\ (3.15) \quad |\nabla^k W_i| &\leq \mathcal{O}(r^{\frac{1}{2} - \frac{k}{2}}) \text{ for each } k \geq 0. \end{aligned}$$

Clearly, for each  $i$ ,  $U_i$  is the unique vector field which satisfies (3.14) and is the extension of a Killing field on  $(\mathcal{S}, \bar{g}_\infty)$ .

We claim that the vector fields  $\{W_i\}_{i=1}^{\bar{d}}$  are linearly independent. To see this, fix a large  $\rho$  and let  $\pi_\rho$  be the pointwise orthogonal projection of vector fields to the hypersurface  $\{f = \rho\}$ . Then

$$(3.16) \quad \pi_\rho(W_i) = \pi_\rho(\tilde{W}_i) = \pi_\rho(U_i) + \pi_\rho(\tilde{W}_i - U_i) = \bar{U}_i + \pi_\rho(\tilde{W}_i - U_i).$$

Moreover, by (3.13) and Lemma 2.7, we have on  $\{f = \rho\}$  the estimate

$$\sup_{\{f=\rho\}} |\pi_\rho(\tilde{W}_i - U_i)|_{\bar{g}_\infty}^2 \leq \frac{C}{\rho} \sup_{\{f=\rho\}} |\pi_\rho(\tilde{W}_i - U_i)|_g^2 = \mathcal{O}(\rho^{-1}).$$

Since  $\bar{U}_i$  are fixed and linearly independent on  $\{f = \rho\}$ , and this is an open condition with respect to the  $C^0(\bar{g}_\infty)$  topology, it follows from (3.16) that  $\{\pi_\rho(W_i)\}_{i=1}^{\bar{d}}$  is linearly independent on  $\{f = \rho\}$  for all large  $\rho$ . Hence,  $\{W_i\}_{i=1}^{\bar{d}}$  is linearly independent on  $M$ , as claimed.

Let  $\mathcal{V}' = \text{span}\{W_i\}_{i=1}^{\bar{d}}$ , so that  $\dim \mathcal{V}' = \bar{d}$  by the last paragraph. Define a map  $\mathcal{R} : \mathcal{V}' \rightarrow \mathfrak{I}(\mathcal{S}, \bar{g}_\infty)$  by sending  $W_i$  to  $\bar{U}_i$  and extending linearly. This map is clearly injective. Also define a map  $\mathcal{T} : \mathcal{V}' \rightarrow \ker_{L^2(e^f)}(L)$  by  $W \mapsto \mathcal{L}_W g$ .

The assumption  $\dim \ker_{L^2(e^f)}(L) \leq d$  implies  $\dim \ker \mathcal{T} \geq \bar{d} - d$ . Let  $\mathcal{V} = \ker \mathcal{T}$ . Then every  $W \in \mathcal{V}$  is a Killing field for  $(M, g)$ . The general identity (3.5) gives  $\Delta W + \text{Ric}(W) = 0$ , so  $[W, \nabla f] = \text{Ric}(W) - \nabla_{\nabla f} W = -\Delta_f W = 0$ . Moreover, we have

$$(3.17) \quad \nabla^2(\mathcal{L}_W f) = \mathcal{L}_W(\nabla^2 f) = \frac{1}{2} \mathcal{L}_W \mathcal{L}_{\nabla f} g = \frac{1}{2} \mathcal{L}_{\nabla f} \mathcal{L}_W g = 0,$$

so the function  $\mathcal{L}_W f = \langle W, \nabla f \rangle$  is constant. Since  $f$  attains a minimum by Lemma 2.1, we conclude that  $\langle W, \nabla f \rangle = 0$  vanishes identically. Thus, we have proved (a), (b), and (c). Part (d) follows from (3.15). Part (e) follows from the uniqueness part of (3.14), and by restricting the map  $\mathcal{R} : \mathcal{V}' \rightarrow \mathfrak{I}(\mathcal{S}, \bar{g}_\infty)$  to  $\mathcal{V}$ .  $\square$

**3.2. Killing fields on asymptotically conical expanding GRSs.** In this subsection, we fix an asymptotically  $(\mathcal{S}, \bar{g}_\infty)$ -conical expanding GRS  $(M^n, g, f)$ . We will prove an analog of Proposition 3.8 in this setting; see Proposition 3.17. The method of proof is essentially the same.

First translate  $f$  upwards so that  $f + \frac{n}{2} > 1$  on  $M$ . Then (2.5) and (2.6) hold with a smaller (perhaps negative) constant  $\mu$ . The following is [12, Lemma 4.1]:

**Lemma 3.9.** *Let  $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ , and let  $Q$  be a vector field on  $M$  such that  $|Q| \leq \mathcal{O}(r^{-2\varepsilon})$ . If a vector field  $V$  satisfies  $(\Delta_f - \frac{1}{2})V = Q$  on  $\{f \leq \rho^2\}$ , then*

$$\sup_{\{f \leq \rho^2\}} \left( |V| - B \left( f + \frac{n}{2} \right)^{-\varepsilon} \right) \leq \max \left\{ \sup_{\{f = \rho^2\}} |V| - B \left( \rho^2 + \frac{n}{2} \right)^{-\varepsilon}, 0 \right\}$$

for some  $B > 0$  independent of  $\rho$ .

*Proof.* Using (2.4), (2.5), and (2.6), we compute

$$\begin{aligned} \left( \Delta_f - \frac{1}{2} \right) \left( f + \frac{n}{2} \right)^{-\varepsilon} &= -\varepsilon \left( f + \frac{n}{2} \right)^{-\varepsilon-1} \Delta_f f - \frac{1}{2} \left( f + \frac{n}{2} \right)^{-\varepsilon} \\ &\quad + \varepsilon(\varepsilon + 1) \left( f + \frac{n}{2} \right)^{-\varepsilon-2} |\nabla f|^2 \\ &= -\varepsilon \left( f + \frac{n}{2} \right)^{-\varepsilon-1} \left( f + \frac{n}{2} + \mu \right) - \frac{1}{2} \left( f + \frac{n}{2} \right)^{-\varepsilon} \end{aligned}$$

$$\begin{aligned}
& + \varepsilon(\varepsilon + 1) \left(f + \frac{n}{2}\right)^{-\varepsilon-2} \left(f + \frac{n}{2} + \mu - R - \frac{n}{2}\right) \\
& = \left(f + \frac{n}{2}\right)^{-\varepsilon} \left(-\varepsilon - \frac{1}{2} + \frac{\varepsilon(\varepsilon + 1) - \varepsilon\mu}{f + n/2} + \frac{\varepsilon(\varepsilon + 1)\mu}{(f + n/2)^2}\right) \\
& \quad - \varepsilon(\varepsilon + 1) \left(R + \frac{n}{2}\right) \left(f + \frac{n}{2}\right)^{-\varepsilon-2} \\
& < -\frac{1}{4} \left(f + \frac{n}{2}\right)^{-\varepsilon},
\end{aligned}$$

where in the last line we took  $\varepsilon > 0$  small. Hence, for large enough  $B > 0$ ,

$$B \left(\Delta_f - \frac{1}{2}\right) \left(f + \frac{n}{2}\right)^{-\varepsilon} < -\frac{B}{4} \left(f + \frac{n}{2}\right)^{-\varepsilon} \leq -|Q|$$

on  $M$ . Also, as in [12, Lemma 4.1], we have  $(\Delta_f - \frac{1}{2})|V| \geq -|Q|$ . Thus,

$$\left(\Delta_f - \frac{1}{2}\right) \left(|V| - B \left(f + \frac{n}{2}\right)^{-\varepsilon}\right) \geq -|Q| + |Q| = 0$$

on  $\{f \leq \rho^2\}$ . The assertion now follows from the maximum principle.  $\square$

The next proposition is [12, Proposition 4.2], the only difference being that we use interior parabolic Schauder estimates rather than a gradient estimate to obtain decay estimates for all derivatives of  $V$ . This is possible given our decay assumptions on  $Q$ .

**Proposition 3.10.** *Let  $Q$  be a vector field on  $M$  such that  $|\nabla^k Q| \leq \mathcal{O}(r^{-\varepsilon-1-k})$  for each  $k \geq 0$ . Then there exists a vector field  $V$  on  $M$  and  $\delta > 0$  satisfying  $(\Delta_f - \frac{1}{2})V = Q$ , with  $|\nabla^k V| \leq \mathcal{O}(r^{-\delta-k})$  for each  $k \geq 0$ .*

We may now establish an analog of Proposition 3.1:

**Proposition 3.11.** *Let  $\bar{U}$  be a KF on  $(S, \bar{g}_\infty)$ , and let  $U$  be a smooth vector field on  $M$  which extends  $\bar{U}$ . Then there is a smooth vector field  $W$  on  $M$  such that*

$$\begin{aligned}
\Delta_f W - \frac{1}{2}W &= 0, \\
|\nabla^k(W - U)| &\leq \mathcal{O}(r^{-\delta-k}) \text{ for each } k \geq 0, \\
|\nabla^k W| &\leq \mathcal{O}(r^{1-k}) \text{ for each } k \geq 0, \\
|\mathcal{L}_W g| &\leq \mathcal{O}(r^{-\delta}),
\end{aligned}$$

for some  $\delta > 0$ .

*Proof.* Since  $(M, g)$  is asymptotically conical, we have

$$|\nabla^k U| \leq \mathcal{O}(r^{1-k}) \text{ for each } k \geq 0.$$

Since  $\mathcal{L}_U g = -\mathcal{L}_U(\phi_* \bar{g} - g) = \nabla U * (\phi_* \bar{g} - g) + U * \nabla(\phi_* \bar{g} - g)$ , this implies by (1.3) that

$$|\nabla^k \mathcal{L}_U g| \leq \mathcal{O}(r^{-\varepsilon-k}) \text{ for each } k \geq 0.$$

Using the general identity (3.5), it follows that

$$(3.18) \quad |\nabla^k(\Delta U + \text{Ric}(U))| \leq C|\nabla^{k+1} \mathcal{L}_U g| \leq \mathcal{O}(r^{-\varepsilon-1-k}) \text{ for each } k \geq 0.$$

Since  $[U, \frac{\nabla f}{|\nabla f|^2}] = 0$  outside a compact set, and we have (2.5), it follows that

$$\nabla_{\nabla f} U - \text{Ric}(U) - \frac{1}{2}U = -[U, \nabla f] = -U(|\nabla f|^2) \frac{\nabla f}{|\nabla f|^2} = \langle U, \nabla R \rangle \frac{\nabla f}{|\nabla f|^2}$$

outside a compact set. This implies

$$|\nabla^k(\nabla_{\nabla_f} U - \text{Ric}(U) - \frac{1}{2}U)| \leq \mathcal{O}(r^{-\varepsilon-1-k}) \quad \text{for all } k \geq 0.$$

Adding this to (3.18) gives

$$|\nabla^k(\Delta_f U - \frac{1}{2}U)| \leq \mathcal{O}(r^{-\varepsilon-1-k}).$$

By Proposition 3.10, there exists a vector field  $V$  satisfying

$$\Delta_f V - \frac{1}{2}V = -\Delta_f U + \frac{1}{2}U$$

on  $M$ , with  $|\nabla^k V| \leq \mathcal{O}(r^{-\delta-k})$  for each  $k$ . Setting  $W := U + V$ , we are done.  $\square$

**Theorem 3.12** ([12, Proposition 3.1]). *Let  $W$  be a vector field satisfying  $\Delta_f W - \frac{1}{2}W = 0$ . Then  $h = \mathcal{L}_W g$  satisfies  $Lh = 0$ .*

As in the previous subsection, we will prove strong decay estimates for solutions of  $Lh = 0$ . This occupies the next four lemmas.

**Lemma 3.13.** *For each  $\ell > 0$ , there exists a compact set  $K_\ell \subset M$  such that for all symmetric 2-tensors  $h$  on  $M$  satisfying  $Lh = 0$ , we have*

$$\Delta_f(f^\ell |h|^2) \geq 0 \quad \text{on } M \setminus K_\ell.$$

*Proof.* Since  $Lh = 0$ , we have

$$\Delta_f |h|^2 = -4\langle \text{Rm}(h), h \rangle + 2|\nabla h|^2.$$

Let  $\ell > 0$ . Mimicking the computation (3.10) but using expander identities, we get

$$\begin{aligned} \Delta_f(f^\ell |h|^2) &\geq \ell f^\ell |h|^2 + \left(\mu + \frac{n}{2}\right) \ell f^{\ell-1} |h|^2 + \ell(\ell-1) f^{\ell-2} |\nabla f|^2 |h|^2 + 2f^\ell |\nabla h|^2 \\ (3.19) \quad &\quad - 4f^\ell |\text{Rm}||h|^2 - 4\ell f^{\ell-1} |\nabla f||h||\nabla h|. \end{aligned}$$

By Lemma 2.9, we have

$$(3.20) \quad |\text{Rm}| \leq C f^{-1}, \quad |\nabla f|^2 \leq C f.$$

Moreover, by the absorbing inequality and Kato's inequality, we have

$$2|\nabla f||h||\nabla h| \leq \frac{f}{\ell} |\nabla h|^2 + \frac{\ell}{f} |\nabla f|^2 |h|^2.$$

Using these facts in (3.19) gives

$$\Delta_f(f^\ell |h|^2) \geq \ell f^\ell |h|^2 + \left(\left(\mu + \frac{n}{2}\right)\ell - C - (\ell + \ell^2)C\right) f^{\ell-1} |h|^2.$$

Since  $f \rightarrow +\infty$  at infinity, this is nonnegative outside a compact set of  $M$ .  $\square$

**Lemma 3.14.** *For each  $\varepsilon \in (0, 1)$ , there exists a compact set  $K_\varepsilon \subset M$  such that for all symmetric 2-tensors  $h$  on  $M$  satisfying  $Lh = 0$ , we have*

$$\Delta_f(e^{\varepsilon f} |h|^2) \geq 0 \quad \text{on } M \setminus K_\varepsilon.$$

*Proof.* Compute

$$\begin{aligned}
\Delta_f(e^{\varepsilon f}|h|^2) &= (\varepsilon e^{\varepsilon f} \Delta_f f + \varepsilon^2 e^{\varepsilon f} |\nabla f|^2) |h|^2 \\
&\quad + e^{\varepsilon f} (2|\nabla h|^2 - 4\langle \text{Rm}(h), h \rangle) + 2\varepsilon e^{\varepsilon f} \langle \nabla f, \nabla |h|^2 \rangle \\
&\geq e^{\varepsilon f} |h|^2 \left( \varepsilon \left( f + \mu + \frac{n}{2} \right) + \varepsilon^2 |\nabla f|^2 \right) + 2e^{\varepsilon f} |\nabla h|^2 \\
(3.21) \quad &\quad - 4e^{\varepsilon f} |\text{Rm}||h|^2 - 4\varepsilon e^{\varepsilon f} |\nabla f||h||\nabla |h||.
\end{aligned}$$

By the absorbing inequality and Kato's inequality, we have

$$2|\nabla f||h||\nabla |h|| \leq \varepsilon^{-1} |\nabla h|^2 + \varepsilon |\nabla f|^2 |h|^2.$$

Using this and (3.20) in (3.21), we obtain

$$\begin{aligned}
\Delta_f(e^{\varepsilon f}|h|^2) &\geq e^{\varepsilon f} |h|^2 \left( \varepsilon \left( f + \mu + \frac{n}{2} \right) - \varepsilon^2 |\nabla f|^2 - C f^{-1} \right) \\
&= e^{\varepsilon f} |h|^2 \left( (\varepsilon - \varepsilon^2) f + \varepsilon \left( \mu + \frac{n}{2} \right) - \varepsilon^2 (\mu - R) - C f^{-1} \right).
\end{aligned}$$

If  $\varepsilon \in (0, 1)$ , then this is nonnegative outside a compact set of  $M$ .  $\square$

**Lemma 3.15.** *Suppose  $u$  is a function such that  $\Delta_f u \geq 0$  outside a compact set and  $u \rightarrow 0$  at infinity. Then  $u \leq C e^{-f}$  on  $M$  for some  $C > 0$ .*

*Proof.* The proof is the same as Lemma 3.6, except that we use the estimate

$$\Delta_f e^{-f} = e^{-f} (-\Delta_f f + |\nabla f|^2) = -e^{-f} \Delta f = -e^{-f} \left( R + \frac{n}{2} \right) \leq 0,$$

since  $R + \frac{n}{2} \geq 0$  on an expanding GRS.  $\square$

**Lemma 3.16.** *If  $Lh = 0$  on  $M$  and  $|h| \leq \mathcal{O}(r^{-\delta})$  for some  $\delta > 0$ , then  $h \in L^2(e^f)$ .*

*Proof.* Let  $\ell < \delta$ , so that  $f^\ell |h|^2 \rightarrow 0$  at infinity. By Lemma 3.13, we have

$$\Delta_f(f^\ell |h|^2) \geq 0 \quad \text{outside a compact set.}$$

Then Lemma 3.15 gives  $f^\ell |h|^2 \leq \mathcal{O}(e^{-f})$ . Hence,  $e^{\frac{1}{2}f} |h|^2 \rightarrow 0$  at infinity. By Lemma 3.14, we have

$$\Delta_f(e^{\frac{1}{2}f} |h|^2) \geq 0 \quad \text{outside a compact set.}$$

Then Lemma 3.15 gives  $|h|^2 \leq \mathcal{O}(e^{-\frac{3}{2}f})$ . This implies  $\int_M |h|^2 e^f < \infty$  since  $(M, g)$  is asymptotically conical.  $\square$

We can now prove an analog of Proposition 3.8. As before,  $\bar{d}$  denotes the dimension of the space of Killing fields on  $(\mathcal{S}, \bar{g}_\infty)$ .

**Proposition 3.17.** *Suppose  $\dim \ker_{L^2(e^f)}(L) \leq d$  for some nonnegative integer  $d$ . Then there exists a  $(\bar{d} - d)$ -dimensional vector space  $\mathcal{V}$  of Killing fields on  $(M, g)$  such that for each  $W \in \mathcal{V}$ ,*

- (a)  $[W, \nabla f] = 0$ .
- (b)  $\langle W, \nabla f \rangle = 0$ , i.e.  $W$  is tangent to the level sets of  $f$ .
- (c)  $W$  is a Killing field for  $(M, g)$  and for each sufficiently far level set of  $f$ .
- (d)  $|\nabla^k W| \leq \mathcal{O}(r^{1-k})$  for each  $k \geq 0$ .
- (e) There is a unique Killing field  $\bar{U}$  on  $(\mathcal{S}, \bar{g}_\infty)$  such that if  $U$  is a smooth vector field on  $M$  which extends  $\bar{U}$ , then  $|\nabla^k(W - U)| \leq \mathcal{O}(r^{-\delta-k})$  for each  $k \geq 0$ . The map  $\mathcal{V} \rightarrow \mathcal{I}(\mathcal{S}, \bar{g}_\infty)$ ,  $W \mapsto \bar{U}$  is linear and injective.

*Proof.* The proof is essentially identical to that of Proposition 3.8, after making the following substitutions: Proposition 3.1  $\mapsto$  Proposition 3.11, Theorem 3.2  $\mapsto$  Theorem 3.12, Lemma 3.7  $\mapsto$  Lemma 3.16, and Lemma 2.7  $\mapsto$  Lemma 2.10.  $\square$

**3.3. Integrating Killing fields to global symmetries.** In this subsection,  $(M^n, g, f)$  is either an asymptotically an  $\mathcal{S}$ -cylindrical steady GRS or an asymptotically  $(\mathcal{S}, \bar{g}_\infty)$ -conical expanding GRS. We also assume that  $(\mathcal{S}, \bar{g}_\infty)$  is homogeneous, meaning its isometry group acts transitively. (In the steady case,  $\bar{g}_\infty$  is understood to be the round metric on  $\mathcal{S}$  of sectional curvature 1.) Our goal here is to use the results from the last two subsections to prove Theorem 1.4.

Let  $\mathcal{I}(M, g)$  be the isometry group of  $(M, g)$  and  $\mathcal{I}^0(M, g)$  its identity component. These are finite-dimensional Lie groups by the Myers–Steenrod theorem. Its Lie algebra  $\mathfrak{I}(M, g)$  is the space of Killing fields.

Let  $\mathcal{V}_\rho$  be the restriction of vector fields in  $\mathcal{V}$  to the level set  $\{f = \rho\}$ . These are Killing fields on  $(\mathcal{S}, g_\rho)$  by Proposition 3.8. Let  $\mathfrak{g}$  be the Lie subalgebra of  $\mathfrak{I}(M, g)$  generated by  $\mathcal{V}$ , and let  $G$  be the unique connected Lie subgroup of  $\mathcal{I}^0(M, g)$  generated by  $\mathfrak{g}$ . We define  $\mathfrak{g}_\rho \subseteq \mathfrak{I}(\mathcal{S}, g_\rho)$  and  $G_\rho \subseteq \mathcal{I}^0(\mathcal{S}, g_\rho)$  analogously.

**Corollary 3.18.** *For any  $X \in \mathfrak{g}$ , we have  $[X, \nabla f] = 0$  and  $\langle X, \nabla f \rangle = 0$ .*

*Proof.* That  $[X, \nabla f] = 0$  follows from Proposition 3.8(a) or Proposition 3.17(a), and the Jacobi identity. Reasoning as in (3.17) gives  $\langle X, \nabla f \rangle = 0$ .  $\square$

For a vector field  $X$  on  $M$ , let  $\Phi_M^X$  be its time 1 flow. If  $(M, g)$  is complete and  $X \in \mathfrak{I}(M, g)$ , then  $\Phi_M^X \in \mathcal{I}^0(M, g)$ .

**Lemma 3.19.** (a) *The restriction map  $\mathcal{V} \rightarrow \mathcal{V}_\rho$  is a linear isomorphism.*

(b) *The restriction map  $\mathfrak{g} \rightarrow \mathfrak{g}_\rho$  is a Lie algebra isomorphism.*

(c) *If  $G_\rho$  is closed in  $\mathcal{I}(\mathcal{S}, g_\rho)$ , then  $G$  is a compact Lie group acting smoothly and isometrically on  $(M, g)$ , and the restriction map  $G \rightarrow G_\rho$  is a Lie group isomorphism.*

*Proof.* Clearly,  $\mathcal{V} \rightarrow \mathcal{V}_\rho$  defines a surjective linear map. For injectivity, suppose  $X, Y \in \mathcal{V}$  agree on  $\{f = \rho\}$ . Then for any point  $x \in \{f = \rho\}$ , we have  $X(x) = Y(x)$  and  $(\nabla X)|_{T_x\{f=\rho\}} = (\nabla Y)|_{T_x\{f=\rho\}}$ . At  $x$ , we also have

$$\nabla_{\nabla f} X - \nabla_{\nabla f} Y = (\text{Ric}(X) - [X, \nabla f]) - (\text{Ric}(Y) - [Y, \nabla f]) = \text{Ric}(X) - \text{Ric}(Y) = 0.$$

Therefore,  $X = Y$  and  $\nabla X = \nabla Y$  at  $x$ . Since  $X$  and  $Y$  are Killing fields for  $(M, g)$ , this implies  $X = Y$  everywhere. Thus, (a) is proved. (b) is proved the same way.

If  $G_\rho$  is closed in  $\mathcal{I}(\mathcal{S}, g_\rho)$ , then since  $\mathcal{I}(\mathcal{S}, g_\rho)$  is compact, so is  $G_\rho$ . Therefore, its exponential map is surjective, i.e.  $G_\rho = \Phi_{\mathcal{S}}^{\mathfrak{g}_\rho}$ . Below, we use this to deduce that  $\Phi_M^{\mathfrak{g}}$  is a subgroup of  $\mathcal{I}^0(M, g)$ , so  $G = \Phi_M^{\mathfrak{g}}$  from the very definition of  $G$ . Then the restriction map  $G \rightarrow G_\rho$  is the obvious map  $\Phi_M^{\mathfrak{g}} \rightarrow \Phi_{\mathcal{S}}^{\mathfrak{g}_\rho}$  which is a group isomorphism. We use this to pull back the smooth structure to  $G$ , and it is easily checked that the usual action  $G \times M \rightarrow M$  is smooth and isometric. This proves (c).

Let  $X, Y \in \mathfrak{g}$ . Let  $X_\rho, Y_\rho \in \mathfrak{g}_\rho$  be their restrictions to  $\{f = \rho\}$ . Since  $G_\rho = \Phi_{\mathcal{S}}^{\mathfrak{g}_\rho}$ , there exists  $Z_\rho \in \mathfrak{g}_\rho$  such that  $\Phi_{\mathcal{S}}^{X_\rho} \circ \Phi_{\mathcal{S}}^{Y_\rho} = \Phi_{\mathcal{S}}^{Z_\rho}$ , and from part (b),  $Z_\rho$  is the restriction of some  $Z \in \mathfrak{g}$ . Thus for all  $x \in \{f = \rho\}$ ,

$$(\Phi_M^X \circ \Phi_M^Y)(x) = (\Phi_{\mathcal{S}}^{X_\rho} \circ \Phi_{\mathcal{S}}^{Y_\rho})(x) = \Phi_{\mathcal{S}}^{Z_\rho}(x) = \Phi_M^Z(x)$$

and by similar reasoning,  $D_x(\Phi_M^X \circ \Phi_M^Y)|_{T_x\{f=\rho\}} = D_x\Phi_M^Z|_{T_x\{f=\rho\}}$ . As before we use Corollary 3.18 to get  $D_x(\Phi_M^X \circ \Phi_M^Y)(\nabla f) = D_x(\Phi_M^Z)(\nabla f) = \nabla f$ . Hence  $\Phi_M^X \circ \Phi_M^Y = \Phi_M^Z$

everywhere on  $M$ . So  $\Phi_M^g$  is closed under composition. The other group properties are easy to check.  $\square$

Next, we will make use of the assumption that  $(\mathcal{S}, \bar{g}_\infty)$  is homogeneous. This implies that its space of Killing fields  $\mathfrak{I}(\mathcal{S}, \bar{g}_\infty)$  spans the tangent space at each point. We define the following quantities:

**Definition 3.20.** Define  $d_1(\mathcal{S}, \bar{g}_\infty) \geq 0$  to be the largest number  $d$  such that

- Any Lie subalgebra of  $\mathfrak{I}(\mathcal{S}, \bar{g}_\infty)$  with dimension  $\geq \bar{d} - d$  spans the tangent space to  $\mathcal{S}$  at every point.

Given  $d' \geq 0$ , we also define  $d_2(\mathcal{S}, d') \geq 0$  to be the largest number  $d$  such that

- If  $g$  is a homogeneous metric on  $\mathcal{S}$  with  $\dim \mathcal{I}(\mathcal{S}, g) \geq \bar{d} - d'$ , then any Lie subalgebra of  $\mathfrak{I}(\mathcal{S}, g)$  with dimension  $\geq \bar{d} - d$  generates a closed Lie subgroup of  $\mathcal{I}(\mathcal{S}, g)$ .

**Lemma 3.21.** If  $\dim \ker_{L^2(\text{ef})}(L) \leq d_1(\mathcal{S}, \bar{g}_\infty)$ , then  $\mathfrak{g}_\rho$  spans the tangent space to  $\{f = \rho\}$  at every point for all sufficiently large  $\rho$ .

*Proof.* We prove the result for asymptotically  $\mathcal{S}$ -cylindrical steady GRS. For asymptotically  $(\mathcal{S}, \bar{g}_\infty)$ -conical expanding GRS, we simply substitute Propositions 3.1 and 3.8 with Propositions 3.11 and 3.17 respectively.

Proposition 3.8 gives a  $(\bar{d} - d_1(\mathcal{S}, \bar{g}_\infty))$ -dimensional space  $\mathcal{V}$  of Killing fields on  $(M, g)$ . Take a basis  $\{W_i\}_{i=1, \dots, \bar{d} - d_1(\mathcal{S}, \bar{g}_\infty)}$  for  $\mathcal{V}$ , let  $\bar{U}_i$  be the corresponding vector fields from Proposition 3.8(e), and let  $U_i$  be arbitrary extensions thereof. Since  $W_i \mapsto \bar{U}_i$  is linear and injective, it follows that  $U_i$  are linearly independent.

Write  $X_i = W_i - U_i$ . From Proposition 3.1 and Proposition 3.8, we have

$$(3.22) \quad |\nabla^k W_i| \leq \mathcal{O}(r^{\frac{1}{2} - \frac{k}{2}}), \quad |\nabla^k X_i| \leq \mathcal{O}(r^{-\frac{k}{2}}), \quad |\nabla^k U_i| \leq \mathcal{O}(r^{\frac{1}{2} - \frac{k}{2}})$$

for each  $k \geq 0$ . We compute

$$X_{ij} := [W_i, W_j] - [U_i, U_j] = [X_i, W_j] + [U_i, X_j] = \nabla(X_i * W_j) + \nabla(U_i * X_j).$$

By (3.22), this yields

$$(3.23) \quad |\nabla^\ell X_{ij}| \leq \mathcal{O}(r^{-\frac{\ell}{2}}) \quad \text{for each } \ell \geq 0.$$

Next, we look at iterated brackets. We have

$$\begin{aligned} X_{ijk} &:= [W_i, [W_j, W_k]] - [U_i, [U_j, U_k]] = [U_i + X_i, [U_j, U_k] + X_{jk}] - [U_i, [U_j, U_k]] \\ &= [U_i, X_{jk}] + [X_i, [U_j, U_k]] + [X_i, X_{jk}] \\ &= \nabla(U_i * X_{jk}) + \nabla(X_i * [U_j, U_k]) + \nabla(X_i * X_{jk}), \end{aligned}$$

which by (3.22) and (3.23), implies

$$|\nabla^\ell X_{ijk}| \leq \mathcal{O}(r^{-\frac{\ell}{2}}) \quad \text{for each } \ell \geq 0.$$

Continuing inductively, we see that each difference

$$X_{i_1 \dots i_p} := [W_{i_1}, [W_{i_2}, [\dots, W_{i_p}]]] - [U_{i_1}, [U_{i_2}, [\dots, U_{i_p}]]]$$

satisfies

$$(3.24) \quad |\nabla^\ell X_{i_1 \dots i_p}| \leq \mathcal{O}(r^{-\frac{\ell}{2}}) \quad \text{for each } \ell \geq 0.$$

From Corollary 3.18, we know that  $X_{i_1 \dots i_p}$  is tangential to the level sets of  $f$ . Then by (3.24) and Lemma 2.7,

$$(3.25) \quad \sup_{\{f=\rho\}} |X_{i_1 \dots i_p}|_{\bar{g}_\infty} \leq \mathcal{O}(\rho^{-\frac{1}{2}}).$$

The vector fields  $\{\bar{U}_i\}_{i=1, \dots, \bar{d}-d_1(\mathcal{S}, \bar{g}_\infty)}$  are linearly independent Killing fields on  $(\mathcal{S}, \bar{g}_\infty)$ . Therefore, the Lie algebra they generate is spanned by a set of the form

$$(3.26) \quad \{[\bar{U}_{i_1}, [\bar{U}_{i_2}, [\dots, \bar{U}_{i_P}]]]\} : 1 \leq i_1, \dots, i_P \leq \bar{d} - d_1(\mathcal{S}, \bar{g}_\infty)\}$$

with  $P < \infty$ . By the definition of  $d_1(\mathcal{S}, \bar{g}_\infty)$ , the vector fields in (3.26) span the tangent space to  $\mathcal{S}$  at each point. Therefore, the corresponding set of extensions

$$\{[U_{i_1}, [U_{i_2}, [\dots, U_{i_P}]]]\} : 1 \leq i_1, \dots, i_P \leq \bar{d} - d_1(\mathcal{S}, \bar{g}_\infty)\}$$

spans the tangent space to  $\{f = \rho\}$  at each point, provided  $\rho$  is sufficiently large. By (3.25),

$$\{[W_{i_1}, [W_{i_2}, [\dots, W_{i_P}]]]\} : 1 \leq i_1, \dots, i_P \leq \bar{d} - d_1(\mathcal{S}, \bar{g}_\infty)\}$$

also spans the tangent space to  $\{f = \rho\}$  at each point, provided  $\rho$  is sufficiently large. Therefore, the subalgebra of Killing fields on  $(M, g)$  generated by the  $W_i$ 's spans the tangent space to  $\{f = \rho\}$  at each point, provided  $\rho$  is sufficiently large.  $\square$

Now we show that Theorem 1.4 holds whenever

$$(3.27) \quad d \leq \min_{d' \leq d_1(\mathcal{S}, \bar{g}_\infty)} \min \{d', d_2(\mathcal{S}, d')\}.$$

*Proof of Theorem 1.4.* Let  $(M^n, g, f)$  be as in the theorem, and suppose  $(\mathcal{S}, \bar{g}_\infty)$  is Riemannian homogeneous. Also assume that  $\dim \ker_{L^2(e^f)}(L) \leq d$ , where  $d$  satisfies (3.27).

Since  $d \leq d_1(\mathcal{S}, \bar{g}_\infty)$ , Lemma 3.21 implies that the subalgebra  $\mathfrak{g}_\rho$  of  $\mathcal{J}(\mathcal{S}, g_\rho)$  spans the tangent space to  $\mathcal{S}$  at every point in any sufficiently far level set of  $f$ . Hence,  $G_\rho$  acts transitively on  $(\mathcal{S}, g_\rho)$ , and  $\dim \mathcal{I}(\mathcal{S}, g_\rho) \geq \dim G_\rho \geq \bar{d} - d \geq \bar{d} - d_1(\mathcal{S}, \bar{g}_\infty)$ . Since  $\mathfrak{g}_\rho$  is a Lie subalgebra of  $\mathcal{J}(\mathcal{S}, g_\rho)$  with  $\dim \mathfrak{g}_\rho \geq \bar{d} - d \geq \bar{d} - d_2(\mathcal{S}, d_1(\mathcal{S}, \bar{g}_\infty))$ , it follows that  $G_\rho$  is a closed Lie subgroup of  $\mathcal{I}(\mathcal{S}, g_\rho)$ . The theorem now follows from Lemma 3.19.  $\square$

Theorem 1.4 as stated is insufficient for applications since it does not tell us what  $G_\rho$  (and hence  $G$ ) is. Later in the paper, we will use specific choices for  $d$  for certain  $(\mathcal{S}, \bar{g}_\infty)$  and rerun the above proof to find all possibilities for  $G$ . In combination with the next result, this will give strong control on the global geometry.

**Proposition 3.22.** *Let  $(M^n, g, f)$  be as in Theorem 1.4, where  $(\mathcal{S}, \bar{g}_\infty)$  is homogeneous. Suppose there is a compact, connected Lie group  $G$  acting smoothly, almost effectively, and isometrically with cohomogeneity one on  $(M, g)$ . Also suppose  $G$  preserves the level sets of  $f$ . Denote the projection map by  $\pi : M \rightarrow M/G$ . Then:*

- (a) *The orbit space  $M/G$  is homeomorphic to  $[0, 1)$ .*
- (b) *There is a closed subgroup  $H$  of  $G$  such that the principal orbits, i.e.  $\pi^{-1}(s)$  for any  $s > 0$ , are diffeomorphic to  $\mathcal{S} \cong G/H$ . Thus, each point in a principal orbit has isotropy subgroup  $H$ .*
- (c) *The singular orbit  $X = \pi^{-1}(0)$  is diffeomorphic to  $G/K$ , where  $K$  is a closed subgroup of  $G$  such that  $H \subset K$  and  $K/H \cong \mathbb{S}^\ell$  for some  $\ell \geq 0$ .*
- (d)  *$M$  is diffeomorphic to the vector bundle  $E = G \times_K \mathbb{R}^{\ell+1}$  over  $G/K \cong X$ , where  $K$  acts via  $k \cdot (g, v) = (gk^{-1}, kv)$ , and  $kv$  is the radial extension of the action of  $K$  on  $K/H \cong \mathbb{S}^\ell$ .*

(e) Every sphere bundle of  $E$  is diffeomorphic to  $\mathcal{S}$ . Thus  $M \setminus X$  is diffeomorphic to  $(0, \infty) \times \mathcal{S}$ , and under this identification the metric  $g$  becomes

$$g = ds^2 + g(s),$$

where  $\{g(s)\}_{s>0}$  is a smooth family of  $G$ -invariant metrics on  $G/H \cong \mathcal{S}$ .

*Proof.* By Mostert [31], the existence of a cohomogeneity one action of a compact Lie group implies that the orbit space  $M/G$  is homeomorphic to one of

$$\mathbb{S}^1, \quad [0, 1], \quad [0, 1), \quad (0, 1).$$

As  $M$  is noncompact, we can rule out  $\mathbb{S}^1$  and  $[0, 1]$ . We can also rule out  $(0, 1)$ , as by [31] this would imply  $M$  is diffeomorphic to  $(0, 1) \times \mathcal{S}$ , contradicting Theorem 2.5. Hence,  $M/G$  is homeomorphic to  $[0, 1)$ , proving (a). The other assertions are well-known facts about manifolds admitting a cohomogeneity one isometric action with orbit space  $[0, 1)$ ; see e.g. [1, §6.3] and [28, §1].  $\square$

#### 4. UNIQUENESS OF ROTATIONALLY SYMMETRIC GRSs

We now specialize the symmetry principles from §3.3 to the case of round spherical asymptotic links, and prove Theorem 1.5. In this section,  $\bar{g}_\infty$  denotes the round metric on  $\mathbb{S}^{n-1}$  of constant sectional curvature 1. Following Theorem 1.5, we take  $(M^n, g, f)$ ,  $n \geq 3$ , to be either an asymptotically  $\mathbb{S}^{n-1}$ -cylindrical steady GRS or an asymptotically  $(\mathbb{S}^{n-1}, \alpha\bar{g}_\infty)$ -conical expanding GRS with  $\alpha > 0$ . We start with a well-known fact (see for instance [29, §II.3]):

**Lemma 4.1.** *The maximal dimension of a proper Lie subalgebra of  $\mathfrak{so}(n)$  is  $\frac{(n-1)(n-2)}{2}$  if  $n = 3$  or  $n \geq 5$ , and 4 if  $n = 4$ .*

Recall the definition of  $d_1$  from Definition 3.20.

**Corollary 4.2.** *We have*

$$(4.1) \quad d_1(\mathbb{S}^{n-1}, \bar{g}_\infty) \geq \begin{cases} n-2 & \text{if } n = 3 \text{ or } n \geq 5, \\ 1 & \text{if } n = 4. \end{cases}$$

*Proof.* Let  $d$  be the quantity on the RHS of (4.1). We have

$$\bar{d} - d = \begin{cases} \frac{n(n-1)}{2} - (n-2) = \frac{(n-1)(n-2)}{2} + 1 & \text{if } n = 3 \text{ or } n \geq 5, \\ 5 & \text{if } n = 4. \end{cases}$$

By Lemma 4.1, any Lie subalgebra of  $\mathfrak{J}(\mathbb{S}^{n-1}, \bar{g}_\infty) = \mathfrak{so}(n)$  with dimension  $\geq \bar{d} - d$  must be  $\mathfrak{so}(n)$  itself, so its vector fields span the tangent space to  $\mathbb{S}^{n-1}$  at each point. Hence,  $d_1(\mathbb{S}^{n-1}, \bar{g}_\infty) \geq d$  by definition.  $\square$

**Corollary 4.3.** *Suppose that*

$$\dim \ker_{L^2(e^f)}(L) \leq \begin{cases} n-2 & \text{if } n = 3 \text{ or } n \geq 5, \\ 1 & \text{if } n = 4. \end{cases}$$

*Then  $SO(n)$  acts smoothly, effectively, and isometrically with cohomogeneity one on  $(M, g)$ . The action preserves the level sets of  $f$ . The isotropy subgroup of each point in a sufficiently far level set of  $f$  is  $SO(n-1)$ .*

*Proof.* The isometry group of the round sphere has dimension  $\bar{d} = \frac{n(n-1)}{2}$ . Let  $d = n - 2$  if  $n = 3$  or  $n \geq 5$ , or  $d = 1$  if  $n = 4$ . Then  $\dim \ker L^2(e^f)(L) \leq d$  by assumption. We rerun the proof of Theorem 1.4:

We have  $d \leq d_1(\mathbb{S}^{n-1}, \bar{g}_\infty)$  by Corollary 4.2. Hence, Lemma 3.21 implies that the subalgebra  $\mathfrak{g}_\rho$  of  $\mathfrak{I}(\mathbb{S}^{n-1}, g_\rho)$  spans the tangent space to  $\mathbb{S}^{n-1}$  at every point in any sufficiently far level set of  $f$ . So  $(\mathbb{S}^{n-1}, g_\rho)$  is Riemannian homogeneous with

$$\dim \mathcal{I}(\mathbb{S}^{n-1}, g_\rho) \geq \bar{d} - d \geq \begin{cases} \frac{(n-1)(n-2)}{2} + 1 & \text{if } n = 3 \text{ or } n \geq 5, \\ 5 & \text{if } n = 4. \end{cases}$$

By Ziller's classification of homogeneous metrics on spheres [38], this is only possible if  $g_\rho$  is round and  $\mathcal{I}^0(\mathbb{S}^{n-1}, g_\rho) = SO(n)$ , and moreover the isotropy subgroup of any point is  $SO(n-1)$ . Since  $\mathfrak{g}_\rho$  is a Lie subalgebra of  $\mathfrak{I}(\mathcal{S}, g_\rho) = \mathfrak{so}(n)$  with  $\dim \mathfrak{g}_\rho \geq \bar{d} - d$ , this forces  $\mathfrak{g}_\rho = \mathfrak{so}(n)$  by Lemma 4.1. Hence  $G_\rho \cong SO(n)$ . By Lemma 3.19,  $G \cong G_\rho$  acts as claimed.  $\square$

**Lemma 4.4.** *Every  $SO(n)$ -invariant metric on  $SO(n)/SO(n-1) \cong \mathbb{S}^{n-1}$  is a multiple of  $\bar{g}_\infty$ .*

*Proof.* Such metrics correspond to  $\text{Ad}(SO(n-1))$ -invariant inner products on  $\mathfrak{p}$ , where  $\mathfrak{so}(n) = \mathfrak{so}(n-1) \oplus \mathfrak{p}$  is a fixed  $\text{Ad}(SO(n-1))$ -invariant decomposition. But  $\mathfrak{p}$  is irreducible, so the result follows from Schur's lemma.  $\square$

*Proof of Theorem 1.5.* By Corollary 4.3 and Proposition 3.22, there is a submanifold  $X \subset M$  not homeomorphic to  $\mathbb{S}^{n-1}$ , such that  $(M \setminus X, g)$  can be identified as

$$(M \setminus X, g) = ((0, \infty) \times \mathbb{S}^{n-1}, ds^2 + g(s)),$$

where  $\{g(s)\}_{s>0}$  is a family of  $SO(n)$ -invariant metrics on a fixed copy of  $SO(n)/SO(n-1) \cong \mathbb{S}^{n-1}$ . From the description of  $(M, g)$  as a bundle over  $X$ , the metric spaces  $(\mathbb{S}^{n-1}, g(s))$  converge in the Hausdorff sense as  $s \rightarrow 0$  to  $X$  with the submanifold metric. Moreover,  $f$  is a function of  $s$ .

By Lemma 4.4, we have

$$(4.2) \quad g(s) = w(s)^2 \bar{g}_\infty \quad \text{for all } s > 0,$$

where  $w$  is a smooth positive function. Therefore, in view of the Hausdorff convergence  $(\mathbb{S}^{n-1}, g(s)) \rightarrow X$  and the fact that  $X$  is not homeomorphic to  $\mathbb{S}^{n-1}$ , we must have  $w(s) \rightarrow 0$  as  $s \rightarrow 0^+$  and  $X$  is a point. By Proposition 3.22(d),  $M$  is diffeomorphic to the cone over  $\mathbb{S}^{n-1}$ , i.e.  $\mathbb{R}^n$ . It is also rotationally symmetric by (4.2).  $\square$

## 5. GLOBAL SYMMETRIES OF GRSs WITH BERGER-TYPE ASYMPTOTIC LINKS

In this section, we specialize the symmetry principles from §3.3 to the case of Berger-type asymptotic links, and prove Theorem 1.6. The basic idea is similar to that of the previous section, but more group-theoretic input is required in this case.

**5.1. Isometries of Berger metrics.** Recall some notations from §1.3:

- For each  $m, k \geq 1$ ,  $L_{m,k}$  is the smooth manifold obtained by quotienting  $\mathbb{S}^{2m+1}$  by the  $\mathbb{Z}_k$  action which rotates the Hopf fibers.
- For  $a, b > 0$ , the metric  $g_{a,b}$  on  $L_{m,k}$  is obtained by quotienting the Berger metric

$$\tilde{g}_{a,b} = a^2 \sigma \otimes \sigma + b^2 \pi^* g_{\text{FS}}$$

on  $\mathbb{S}^{2m+1}$  by the  $\mathbb{Z}_k$  action. The metric  $g_{a,b}$  has constant sectional curvature  $\kappa^{-2}$  if and only if  $a = b = \kappa$ .

The next two lemmas lay out well-known facts about the isometry groups of  $\tilde{g}_{a,b}$  and  $g_{a,b}$ , as well as their homogeneous presentations.

**Lemma 5.1.** *Let  $m, k \geq 1$  and  $a, b > 0$ .*

- (a)  $U(m+1)$  embeds into  $\mathcal{I}^0(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})$  as a transitive subgroup of isometries, with isotropy subgroup  $U(m)$  (embedded in  $U(m+1)$  as  $A \mapsto \text{diag}(A, 1)$ ).
- (b)  $SU(m+1)$  embeds into  $\mathcal{I}^0(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})$  as a transitive subgroup of isometries, with isotropy subgroup  $SU(m)$  (embedded in  $SU(m+1)$  as  $A \mapsto \text{diag}(A, 1)$ ).
- (c)  $(L_{m,k}, g_{a,b})$  is the quotient of  $(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})$  by a central subgroup  $\mathbb{Z}_k \subset U(1) \cong Z(U(m+1))$ .
- (d)  $U(m+1)$  acts transitively and isometrically on  $(L_{m,k}, g_{a,b})$  with isotropy subgroup  $U(m) \times \mathbb{Z}_k$  (block-diagonally embedded in  $U(m+1)$ ).
- (e)  $SU(m+1)$  acts transitively and isometrically on  $(L_{m,k}, g_{a,b})$  with isotropy subgroup  $S(U(m) \times \mathbb{Z}_k)$  (block-diagonally embedded in  $SU(m+1)$ ).

*Proof.* Parts (a), (b), and (c) are standard. Part (d) follows from (a) and (c) because the subgroup of  $U(m+1)$  generated by the upper-left  $U(m)$  and the central  $\mathbb{Z}_k$  is the indicated copy of  $U(m) \times \mathbb{Z}_k$ . Part (e) follows from parts (b) and (d).  $\square$

**Lemma 5.2.** *Let  $m \geq 1$  and  $a, b > 0$ . Then*

$$\mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g}_{a,b}) \cong \begin{cases} O(2m+2) & \text{if } a = b, \\ U(m+1) \rtimes \mathbb{Z}_2 & \text{if } a \neq b. \end{cases}$$

If additionally  $k \geq 3$ , then for any  $a, b > 0$  we have

$$(5.1) \quad \mathcal{I}(L_{m,k}, g_{a,b}) \cong (U(m+1) \rtimes \mathbb{Z}_2) / \mathbb{Z}_k.$$

*Proof.* If  $a = b$ , then  $(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})$  is a round sphere with isometry group  $O(2m+2)$ . If  $a \neq b$ , it is well-known (e.g. [35, §2]) that  $\mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g}_{a,b}) = \hat{\mathcal{I}}$  where

$$\hat{\mathcal{I}} = \{A \in O(2m+2) : AJ = \pm JA\},$$

and  $J \in O(2m+2)$  is the matrix associated with multiplication by  $i$  in  $\mathbb{C}^{m+1} \cong \mathbb{R}^{2m+2}$ . One checks that  $\hat{\mathcal{I}}$  is the disjoint union

$$(5.2) \quad \hat{\mathcal{I}} = U(m+1) \sqcup KU(m+1),$$

where  $U(m+1)$  is considered as embedded in  $O(2m+2)$ , and  $K \in O(2m+2)$  is the matrix which sends  $(z_0, \dots, z_m) \in \mathbb{C}^{m+1} \cong \mathbb{R}^{2m+2}$  to  $(\bar{z}_0, \dots, \bar{z}_m)$ . Then

$$(5.3) \quad \hat{\mathcal{I}} \cong U(m+1) \rtimes \mathbb{Z}_2.$$

By Lemma 5.1(c), we have  $\mathcal{I}(L_{m,k}, g_{a,b}) \cong N_{\mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})}(\mathbb{Z}_k) / \mathbb{Z}_k$ , where  $\mathbb{Z}_k$  sits centrally in  $U(m+1) \subset \mathcal{I}^0(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})$ . Thus, to prove (5.1) we must show that  $N_{\mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})}(\mathbb{Z}_k) \cong U(m+1) \rtimes \mathbb{Z}_2$ . We again divide into two cases.

Suppose  $a \neq b$ . With  $K \in O(2m+2)$  as above, we see that  $KAK^{-1} = \bar{A}$  for all  $A \in U(m+1) \subset O(2m+2)$ . Thus,  $K$  normalizes the indicated central copy of  $\mathbb{Z}_k$ . Using (5.2) and (5.3), it follows that

$$N_{\mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g}_{a,b})}(\mathbb{Z}_k) = U(m+1) \sqcup KU(m+1) \cong U(m+1) \rtimes \mathbb{Z}_2,$$

as required.

On the other hand, suppose  $a = b$ , so that  $\mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g}_{a,b}) = O(2m+2)$ . The matrix  $Z = \exp\left(\frac{2\pi}{k}J\right)$  is a generator of the indicated copy of  $\mathbb{Z}_k$ . Thus,

$$(5.4) \quad Z^{\pm 1} = \cos\left(\frac{2\pi}{k}\right)I \pm \sin\left(\frac{2\pi}{k}\right)J,$$

where  $I$  is the identity matrix of size  $2m+2$ . If  $A \in N_{O(2m+2)}(\mathbb{Z}_k)$ , then  $AZA^{-1} = Z^\alpha$  for some  $\alpha \in \mathbb{Z}$ , i.e.

$$A \exp\left(\frac{2\pi}{k}J\right)A^{-1} = \exp\left(\frac{2\pi\alpha}{k}J\right).$$

The left matrix has eigenvalues  $e^{\pm \frac{2\pi i}{k}}$ , each with multiplicity  $m+1$ . The right matrix has eigenvalues  $e^{\pm \frac{2\pi i\alpha}{k}}$ , each with multiplicity  $m+1$ . For these eigenvalues to match, we need  $\alpha \equiv \pm 1 \pmod{k}$ . Then  $AZA^{-1} = Z^{\pm 1}$ . Writing these two equations out in terms of (5.4), then simplifying, we get

$$\sin\left(\frac{2\pi}{k}\right)(AJ - JA) = 0 \quad \text{or} \quad \sin\left(\frac{2\pi}{k}\right)(AJ + JA) = 0.$$

Since  $k \geq 3$ , we have  $\sin\left(\frac{2\pi}{k}\right) \neq 0$ , so  $AJ = \pm JA$  and  $A \in \hat{\mathcal{I}}$ . Thus,  $N_{O(2m+2)}(\mathbb{Z}_k) \subseteq \hat{\mathcal{I}} \cong U(m+1) \rtimes \mathbb{Z}_2$ . Reversing these arguments gives  $\hat{\mathcal{I}} \subseteq N_{O(2m+2)}(\mathbb{Z}_k)$ .  $\square$

**5.2. Reduction to cohomogeneity one.** The main result of this subsection is Corollary 5.6, which is an analog of Corollary 4.3 from the previous section. We need the following preparations.

**Lemma 5.3.** *Let  $m \geq 1$ . If  $\mathfrak{g}$  is a Lie subalgebra of  $\mathfrak{u}(m+1)$  satisfying*

$$(5.5) \quad \dim \mathfrak{g} \geq (m+1)^2 - 2(m+1) + 3,$$

*Then either  $\mathfrak{g} = \mathfrak{su}(m+1)$  or  $\mathfrak{g} = \mathfrak{u}(m+1)$ . If  $m = 3$  we also allow  $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$  (up to conjugacy in  $\mathfrak{u}(4)$ ).*

*Proof.* Consider the canonical representation of  $\mathfrak{g}$  on  $\mathbb{C}^{m+1}$ . If this were reducible, then  $\mathfrak{g}$  is contained in  $\mathfrak{u}(k) \oplus \mathfrak{u}(m+1-k)$  for some  $1 \leq k \leq m$ . However, a dimensionality argument shows that this contradicts (5.5). Therefore,  $\mathfrak{g}$  acts irreducibly on  $\mathbb{C}^{m+1}$ .

Recall that  $\mathfrak{u}(m+1) = \mathfrak{su}(m+1) \oplus \mathfrak{u}(1)$ , where  $\mathfrak{u}(1) = \mathbb{R}iI_{m+1}$ . Thus,

$$(5.6) \quad \mathfrak{g} = \mathfrak{g}^{\text{tf}} \oplus \mathfrak{z} := (\mathfrak{g} \cap \mathfrak{su}(m+1)) \oplus (\mathfrak{g} \cap \mathfrak{u}(1)).$$

Since  $\dim \mathfrak{z} \leq 1$ , we have  $\dim \mathfrak{g}^{\text{tf}} \geq \dim \mathfrak{g} - 1$ . Moreover, since  $\mathfrak{g}^{\text{tf}}$  is a subalgebra of  $\mathfrak{su}(m+1)$ , it is compact and hence reductive. Also  $\mathfrak{g}^{\text{tf}}$  acts irreducibly on  $\mathbb{C}^{m+1}$ , since  $\mathfrak{g}$  does. Hence, its complexification  $\mathfrak{g}_{\mathbb{C}}^{\text{tf}} \subset \mathfrak{sl}(m+1, \mathbb{C})$  is also reductive, acts irreducibly on  $\mathbb{C}^{m+1}$ , and satisfies

$$(5.7) \quad \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{\text{tf}} \geq (m+1)^2 - 2(m+1) + 2.$$

By a dimensional argument, one checks that the canonical representation of  $\mathfrak{g}_{\mathbb{C}}^{\text{tf}}$  on  $\mathbb{C}^{m+1}$  does not decompose as a tensor product. Combined with its reductivity, this implies  $\mathfrak{g}_{\mathbb{C}}^{\text{tf}}$  is a simple Lie algebra over  $\mathbb{C}$  possessing an faithful  $\mathbb{C}$ -irrep of dimension  $m+1$ . Also being a subalgebra of  $\mathfrak{sl}(m+1, \mathbb{C})$  and satisfying (5.7), one checks using the classification of simple Lie algebras and their smallest  $\mathbb{C}$ -irreps that the only possibilities are  $\mathfrak{g}_{\mathbb{C}}^{\text{tf}} = \mathfrak{sl}(m+1, \mathbb{C})$ , and if  $m = 3$ ,  $\mathfrak{g}_{\mathbb{C}}^{\text{tf}} = \mathfrak{sp}(2, \mathbb{C})$ .

If  $\mathfrak{g}_{\mathbb{C}}^{\text{tf}} = \mathfrak{sl}(m+1, \mathbb{C})$ , then  $\mathfrak{g}^{\text{tf}}$  is a compact real form of  $\mathfrak{sl}(m+1, \mathbb{C})$ , which is necessarily  $\mathfrak{su}(m+1)$ . Using (5.6), we see that  $\mathfrak{g} = \mathfrak{su}(m+1)$  if  $\dim \mathfrak{z} = 0$ , or  $\mathfrak{g} = \mathfrak{u}(m+1)$  if  $\dim \mathfrak{z} = 1$ .

If  $m = 3$  and  $\mathfrak{g}_{\mathbb{C}}^{\text{tf}} = \mathfrak{sp}(2, \mathbb{C})$ , then  $\mathfrak{g}^{\text{tf}}$  is a compact real form of  $\mathfrak{sp}(2, \mathbb{C})$ , which is necessarily  $\mathfrak{sp}(2)$ . Using (5.6), we see that  $\mathfrak{g} = \mathfrak{sp}(2)$  if  $\dim \mathfrak{z} = 0$ , or  $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$  if  $\dim \mathfrak{z} = 1$ . We can rule out the former case since this violates (5.5).  $\square$

**Proposition 5.4.** *Let  $m \geq 1$ ,  $k \geq 3$ , and  $a, b > 0$ . Then*

$$d_1(L_{m,k}, g_{a,b}) \geq 2(m+1) - 3.$$

*Proof.* Let  $\mathfrak{g}$  be a Lie subalgebra of  $\mathfrak{J}(L_{m,k}, g_{a,b}) = \mathfrak{u}(m+1)$  with dimension  $\geq \bar{d} - 2(m+1) + 3 = (m+1)^2 - 2(m+1) + 3$ . By Lemma 5.3, either  $\mathfrak{g} = \mathfrak{su}(m+1)$  or  $\mathfrak{g} = \mathfrak{u}(m+1)$ , or when  $m = 3$  we may have  $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ . Therefore, the unique connected Lie subgroup  $G$  of  $\mathcal{I}^0(L_{m,k}, g_{a,b}) = U(m+1)/\mathbb{Z}_k$  with Lie algebra  $\mathfrak{g}$  is, up to conjugacy, the image of  $\tilde{G} \in \{SU(m+1), U(m+1), Sp(2) \cdot U(1)\}$  under the projection map  $U(m+1) \rightarrow U(m+1)/\mathbb{Z}_k$ . Note that each possible  $\tilde{G}$  acts transitively on the unit (coordinate) sphere in  $\mathbb{C}^{m+1}$ , so  $G$  acts transitively on  $L_{m,k}$ . Thus,  $\mathfrak{g}$  spans the tangent space to  $L_{m,k}$  at each point. This implies that  $d_1(L_{m,k}, g_{a,b}) \geq 2(m+1) - 3$ .  $\square$

**Proposition 5.5.** *Let  $m \geq 1$  and  $k \geq 3$ . If  $g$  is a homogeneous metric on  $L_{m,k}$  with*

$$(5.8) \quad \dim \mathcal{I}(L_{m,k}, g) \geq \begin{cases} (m+1)^2 - 2(m+1) + 3 & \text{if } m = 2 \text{ or } m \geq 4, \\ 14 & \text{if } m = 3, \\ 4 & \text{if } m = 1, \end{cases}$$

*then  $g$  is a Berger metric,  $\mathcal{I}^0(L_{m,k}, g) = U(m+1)/\mathbb{Z}_k$ , and the isotropy subgroup of its action is the image of  $U(m) \times \mathbb{Z}_k$  under the quotient map  $U(m+1) \rightarrow U(m+1)/\mathbb{Z}_k$ .*

*Proof.* Let  $(\mathbb{S}^{2m+1}, \tilde{g})$  be the universal cover of  $(L_{m,k}, g)$ . Then  $\dim \mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g})$  has the same lower bound as in (5.8). Meanwhile, Ziller [38] classifies all homogeneous metrics on spheres and their isometry groups. Using this classification and the dimension bound on  $\mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g})$ , all non-Berger cases are ruled out so  $\tilde{g}$  must be a Berger metric.

If  $\tilde{g}$  has constant sectional curvature, then since diffeomorphic spherical space forms are isometric [21],  $(L_{m,k}, g)$  is isometric to  $(L_{m,k}, \bar{g}_{\infty})$ . The remaining claims then follow from Lemma 5.2.

Suppose now that  $\tilde{g}$  has non-constant sectional curvature. By construction,  $(L_{m,k}, g)$  is the quotient of  $(\mathbb{S}^{2m+1}, \tilde{g})$  by a free, isometric action of  $\mathbb{Z}_k$ . By (5.8), there is an embedding  $\iota : \mathbb{Z}_k \rightarrow G := \mathcal{I}(\mathbb{S}^{2m+1}, \tilde{g}) = U(m+1) \rtimes \mathbb{Z}_2$  with the same dimension lower bound on  $H := N_G^0(\iota(\mathbb{Z}_k))$ . Let  $F := \iota(\mathbb{Z}_k)$ . Consider the conjugation map  $H \rightarrow \text{Aut}(F)$ . This has connected image in a finite group, so it is trivial and hence

$$(5.9) \quad F \subset C_G(H).$$

Now,  $H$  is a closed connected subgroup of  $G$  with dimension at least the right-hand side of (5.8). By Lemma 5.3, its Lie algebra is  $\mathfrak{su}(m+1)$  or  $\mathfrak{u}(m+1)$ , with the former only possible when  $m > 1$  due to the dimension bound. Accordingly,  $H = SU(m+1)$  or  $H = U(m+1)$ . Below, we show that in both cases  $F = \iota(\mathbb{Z}_k)$  is contained in the center of  $\mathcal{I}^0(\mathbb{S}^{2m+1}, \tilde{g}) = U(m+1)$ . This implies that  $g$  is a Berger metric, and the remaining claims of the proposition follow from Lemma 5.2.

**Case 1:**  $H = U(m+1)$ . Suppose  $g \in KU(m+1)$  centralizes  $H$ . Then it commutes with all elements of the form  $zI_{m+1}$ ,  $z \in U(1)$ . Writing  $g = KA$  for  $A \in U(m+1)$ , we have

$$zg = (zI_{m+1})g = g(zI_{m+1}) = KAz = KzA = \bar{z}KA = \bar{z}g.$$

for all  $z \in U(1)$ . This is a contradiction, so no element in  $KU(m+1)$  centralizes  $H$ . Thus,

$$F \subset C_G(H) = C_{U(m+1) \sqcup KU(m+1)}(H) = Z(U(m+1)).$$

**Case 2:**  $H = SU(m+1)$  (only when  $m > 1$ ). It is well known that

$$(5.10) \quad C_{U(m+1)}(H) = Z(U(m+1)) \cong U(1).$$

Meanwhile, let  $g \in KU(m+1)$ , and write  $g = KA$  with  $A \in U(m+1)$ . For all  $S \in H$ , we have

$$gS = Sg \iff KAS = S(KA) = K\bar{S}A \iff ASA^{-1} = \bar{S}.$$

That is,  $g = KA$  centralizes  $H$  iff  $ASA^{-1} = \bar{S}$  for all  $S \in H$ . This can only happen if  $S \mapsto \bar{S}$  is an inner automorphism of  $SU(m+1)$ . However, since  $m > 1$ , complex conjugation in  $SU(m+1)$  is not inner. Therefore, no element of  $KU(m+1)$  centralizes  $H$ . Together with (5.9) and (5.10), we conclude that  $F \subset C_G(H) = Z(U(m+1))$ , as claimed.  $\square$

**Corollary 5.6.** *Let  $(M, g, f)$  be an asymptotically  $L_{m,k}$ -cylindrical steady GRS or an asymptotically  $(L_{m,k}, g_{\alpha,\beta})$ -conical expanding GRS, where  $m \geq 1$ ,  $k \geq 3$ , and  $\alpha, \beta > 0$ . Suppose that*

$$(5.11) \quad \dim \ker_{L^2(e^f)}(L) \leq \begin{cases} 2m-1 & \text{if } m=2 \text{ or } m \geq 4, \\ 2 & \text{if } m=3, \\ 0 & \text{if } m=1. \end{cases}$$

*Let  $G = SU(m+1)$  if  $m \geq 2$ , and  $G = U(2)$  if  $m = 1$ . Then  $G$  acts smoothly, almost effectively, and isometrically with cohomogeneity one on  $(M, g)$ . The action preserves the level sets of  $f$ . The isotropy subgroup of each point in a sufficiently far level set of  $f$  is  $S(U(m) \times \mathbb{Z}_k)$  if  $m \geq 2$ , and  $U(1) \times \mathbb{Z}_k$  if  $m = 1$ .*

*Proof.* We have  $\bar{d} = (m+1)^2$ . Let  $d$  be the number on the right-hand side of (5.11). We rerun the proof of Theorem 1.4:

By Proposition 5.4,  $\dim \ker_{L^2(e^f)}(L) \leq d \leq d_1(L_{m,k}, \bar{g}_\infty)$  (or  $d_1(L_{m,k}, g_{\alpha,\beta})$  in the expanding case). Hence, Lemma 3.21 implies that the subalgebra  $\mathfrak{g}_\rho$  of  $\mathfrak{J}(L_{m,k}, g_\rho)$  spans the tangent space to  $L_{m,k}$  at any point in any sufficiently far level set of  $f$ . Hence,  $(L_{m,k}, g_\rho)$  is Riemannian homogeneous with

$$\dim \mathcal{I}(L_{m,k}, g_\rho) \geq \bar{d} - d \geq \begin{cases} (m+1)^2 - 2(m+1) + 3 & \text{if } m=2 \text{ or } m \geq 4 \\ 14 & \text{if } m=3, \\ 4 & \text{if } m=1. \end{cases}$$

By Proposition 5.5,  $g_\rho$  is a Berger metric, and  $\mathcal{I}^0(L_{m,k}, g_\rho) = U(m+1)/\mathbb{Z}_k$  acts transitively with isotropy subgroup described in the proposition. Since  $\mathfrak{g}_\rho$  is a Lie subalgebra of  $\mathfrak{J}(L_{m,k}, g_\rho) = \mathfrak{u}(m+1)$  with  $\dim \mathfrak{g}_\rho \geq \bar{d} - d$ , Lemma 5.3 implies that  $\mathfrak{g}_\rho \in \{\mathfrak{su}(m+1), \mathfrak{u}(m+1)\}$  if  $m \geq 2$ , or  $\mathfrak{g}_\rho = \mathfrak{u}(m+1)$  if  $m = 1$ . Hence, for the subgroup  $G_\rho$  of  $\mathcal{I}^0(L_{m,k}, g_\rho)$  that  $\mathfrak{g}_\rho$  integrates to, we know the following:

- If  $m \geq 2$ , then  $G_\rho$  is the image of either  $SU(m+1)$  or  $U(m+1)$  under the quotient map  $U(m+1) \rightarrow U(m+1)/\mathbb{Z}_k$ . In the former case,  $G_\rho$  still acts transitively on  $(L_{m,k}, g_\rho)$  with isotropy subgroup given by the image of  $S(U(m) \times \mathbb{Z}_k)$  under the same quotient map.
- If  $m = 1$ , then  $G_\rho \cong U(2)/\mathbb{Z}_k$ .

The corollary now follows from Lemma 3.19, and by undoing the  $\mathbb{Z}_k$ -quotients in the groups involved. (This last step leads to the action being almost effective instead of effective.)  $\square$

**5.3. Proof of Theorem 1.6.** Having now pinned down the cohomogeneity one structure due to Corollary 5.6, in this final part we sharpen this to Theorem 1.6. In particular, we need to reduce the metric to an explicit system of ODEs.

**Lemma 5.7.** (a) *Let  $m \geq 2$ . If  $K$  is a closed proper subgroup of  $SU(m+1)$  that contains  $SU(m)$ , then  $K = S(U(m) \times C)$  for some subgroup  $C$  of  $U(1)$ .*  
 (b) *Meanwhile, if  $K$  is a closed proper subgroup of  $U(2)$  that contains  $U(1)$ , then either  $K = U(1) \times C$  or  $K = (U(1) \times U(1)) \rtimes \mathbb{Z}_2$ .*

*Proof.* We focus on case (a). The identity component  $K^0$  of  $K$  acts canonically on  $\mathbb{C}^{m+1}$ , and splits  $\mathbb{C}^{m+1}$  into irreducible modules which are orthogonal with respect to the standard Hermitian metric. As  $K^0$  contains the upper-left  $SU(m)$ , there are two possibilities:

- (i) The decomposition is  $\mathbb{C}^m \oplus \mathbb{C}^1$ .
- (ii) The decomposition is  $\mathbb{C}^{m+1}$ , i.e.  $K^0$  acts irreducibly.

**Case (i):** in this case,  $K^0$  contains the upper-left  $SU(m)$  and is a subgroup of

$$\{g \in SU(m+1) : g(\mathbb{C}^m) = \mathbb{C}^m, g(\mathbb{C}^1) = \mathbb{C}^1\} = S(U(m) \times U(1)).$$

Thus  $K^0 = S(U(m) \times C)$  for some subgroup  $C$  of  $U(1)$ . As  $K^0$  is connected, the only possibilities are  $K^0 = SU(m)$  or  $K^0 = S(U(m) \times U(1))$ . Since  $K^0$  is a normal subgroup of  $K$ , we have  $K \subseteq N_{SU(m+1)}(K^0)$ . Using this, we find the possibilities for  $K$ :

- If  $K^0 = SU(m)$ , then  $N_{SU(m+1)}(K^0) = S(U(m) \times U(1))$  so  $K$  is of the form  $S(U(m) \times C)$ .
- If  $K^0 = S(U(m) \times U(1))$ , then  $N_{SU(m+1)}(K^0) = K^0$ , so  $K = K^0$ .

**Case (ii):** since  $K^0$  contains  $SU(m)$ , its Lie algebra  $\mathfrak{k} \subseteq \mathfrak{su}(m+1)$  contains  $\mathfrak{su}(m)$ , embedded into the upper-left block. Since  $K^0$  is connected and acts irreducibly on  $\mathbb{C}^{m+1}$ , the induced representation  $\mathfrak{k}$  of  $\mathbb{C}^{m+1}$  is also irreducible. Hence, there exists  $X \in \mathfrak{k}$  with nonzero off-diagonal entries with respect to the vector space decomposition  $\mathbb{C}^{m+1} = \mathbb{C}^m \oplus \mathbb{C}^1$ , i.e.

$$X = \begin{bmatrix} A & u \\ -u^* & \alpha \end{bmatrix}$$

where  $A \in \mathfrak{u}(m)$ ,  $u \in \mathbb{C}^{m \times 1}$ , and  $\alpha \in i\mathbb{R}$ , with  $\text{diag}(A, \alpha) \in \mathfrak{su}(m+1)$  and  $u \neq 0$ . Using this and the fact that  $\mathfrak{k}$  contains  $\mathfrak{su}(m)$ , it can be checked by computing commutators that  $\mathfrak{k} = \mathfrak{su}(m+1)$ . We omit the details. It follows that  $K^0 = SU(m+1)$ , and  $K = SU(m+1)$ .

This proves part (a) of the lemma. Part (b) is proved analogously. The only substantive difference lies in case (i), where in the case  $K^0 = U(1) \times U(1)$  we get  $N_{U(2)}(K^0) = K^0 \rtimes \mathbb{Z}_2$ , so either  $K = K^0$  or  $K = K^0 \rtimes \mathbb{Z}_2$ .  $\square$

**Corollary 5.8.** (a) *Let  $m \geq 2$  and  $k \geq 1$ . If  $K$  is a closed subgroup of  $SU(m+1)$  containing  $S(U(m) \times \mathbb{Z}_k)$ , and  $K/S(U(m) \times \mathbb{Z}_k)$  is diffeomorphic to  $\mathbb{S}^\ell$  for some  $\ell \geq 0$ , then  $K = S(U(m) \times \mathbb{Z}_{2k})$  or  $K = S(U(m) \times U(1))$ .*

(b) *Let  $k \geq 1$ . If  $K$  is a closed subgroup of  $U(2)$  containing  $U(1) \times \mathbb{Z}_k$ , and  $K/(U(1) \times \mathbb{Z}_k)$  is diffeomorphic to  $\mathbb{S}^\ell$  for some  $\ell \geq 0$ , then  $K = U(1) \times \mathbb{Z}_{2k}$  or  $K = U(1) \times U(1)$ .*

By Lemma 5.1, there are diffeomorphisms  $SU(m+1)/S(U(m) \times \mathbb{Z}_k) \cong L_{m,k}$  and  $U(2)/(U(1) \times \mathbb{Z}_k) \cong L_{1,k}$ . We fix such an identification for the next lemma, which is an analog of Lemma 4.4.

**Lemma 5.9.** (a) *Let  $m \geq 2$  and  $k \geq 1$ . Then any  $SU(m+1)$ -invariant metric on  $SU(m+1)/S(U(m) \times \mathbb{Z}_k) \cong L_{m,k}$  is equal to  $g_{a,b}$  for some  $a, b > 0$ .*

(b) Let  $k \geq 1$ . Any  $U(2)$ -invariant metric on  $U(2)/(U(1) \times \mathbb{Z}_k) \cong L_{1,k}$  is equal to  $g_{a,b}$  for some  $a, b > 0$ .

*Proof.* We handle case (a); case (b) is entirely similar.

Let  $G = SU(m+1)$  and  $H = S(U(m) \times \mathbb{Z}_k)$ . Then  $G$ -invariant metrics on  $G/H$  bijectively correspond to  $\text{Ad}(H)$ -invariant inner products on  $\mathfrak{p}$ , where  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  is a fixed  $\text{Ad}(H)$ -invariant decomposition. It is easy to see that  $\mathfrak{p}$  further decomposes into irreducible  $\text{Ad}(H)$ -modules  $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ , with  $\dim_{\mathbb{R}} \mathfrak{p}_1 = 1$  and  $\dim_{\mathbb{R}} \mathfrak{p}_2 = 2m$ . By Schur's lemma, all  $\text{Ad}(H)$ -invariant inner products on  $\mathfrak{p}$  are of the form

$$a^2 \langle \cdot, \cdot \rangle_{\mathfrak{p}_1} + b^2 \langle \cdot, \cdot \rangle_{\mathfrak{p}_2},$$

where  $a, b > 0$  and  $\langle \cdot, \cdot \rangle_{\mathfrak{p}_i}$  is the Euclidean inner product on  $\mathfrak{p}_i$ . Tracing back identifications, this corresponds to the metric  $g_{a,b}$  on  $L_{m,k}$ . The lemma follows.  $\square$

We finally proceed to the proof of our main result.

*Proof of Theorem 1.6.* Let  $(M, g, f)$  be an asymptotically  $L_{m,k}$ -cylindrical steady GRS or an asymptotically  $(L_{m,k}, g_{\alpha,\beta})$ -conical expanding GRS, with  $m \geq 1$ ,  $k \geq 3$ , and  $\alpha, \beta > 0$ . Suppose that

$$\dim \ker_{L^2(e^f)}(L) \leq \begin{cases} 2m-1 & \text{if } m=2 \text{ or } m \geq 4, \\ 2 & \text{if } m=3, \\ 0 & \text{if } m=1. \end{cases}$$

By Corollary 5.6 and Proposition 3.22, we get the following.

(i) Outside a submanifold  $X$ , we can identify

$$(M \setminus X, g) = ((0, \infty) \times G/H, ds^2 + g(s)),$$

where  $\{g(s)\}_{s>0}$  are  $G$ -invariant metrics on a fixed copy of  $G/H \cong L_{m,k}$ . The groups  $G$  and  $H$  are given by  $G = SU(m+1)$ ,  $H = S(U(m) \times \mathbb{Z}_k)$  if  $m \geq 2$ , and  $G = U(2)$ ,  $H = U(1) \times \mathbb{Z}_k$  if  $m = 1$ .

- (ii)  $f$  is a function of  $s$ , and  $f$  is constant on  $X$ .
- (iii) There is a closed subgroup  $K$  of  $G$  containing  $H$  such that  $K/H \cong \mathbb{S}^\ell$  for some  $\ell \geq 0$ , and  $X \cong G/K$ .
- (iv) From the description of  $(M, g)$  as a fiber bundle over  $X$ , the metric spaces  $(G/H, g(s))$  converge in the Hausdorff sense to  $X$  as  $s \rightarrow 0$ .

From (i) and Lemma 5.9, we have  $g(s) = g_{a(s), b(s)}$  for some positive functions  $a, b : (0, \infty) \rightarrow \mathbb{R}$ . Combining this with (ii) and the computations in [2, Appendix A], the triple  $(f, a, b)$  satisfies (1.6)–(1.8). Moreover, if  $(M, g, f)$  is asymptotically  $L_{m,k}$ -cylindrical, then

$$(5.12) \quad \lim_{s \rightarrow \infty} \frac{a(s)}{b(s)} = 1,$$

and if  $(M, g, f)$  is asymptotically  $(L_{m,k}, g_{\alpha,\beta})$ -conical, then

$$(5.13) \quad \lim_{s \rightarrow \infty} \frac{a(s)}{s} = \alpha, \quad \lim_{s \rightarrow \infty} \frac{b(s)}{s} = \beta.$$

It remains to examine the boundary conditions. By (iii) and Corollary 5.8, there are two possibilities for  $K$  and  $X$ :

- (a)  $K = S(U(m) \times \mathbb{Z}_{2k})$  (if  $m \geq 2$ ) or  $K = U(1) \times \mathbb{Z}_{2k}$  (if  $m = 1$ ), and  $X \cong L_{m,2k}$ .
- (b)  $K = S(U(m) \times U(1))$  (if  $m \geq 2$ ) or  $K = U(1) \times U(1)$  (if  $m = 1$ ), and  $X \cong \mathbb{C}\mathbb{P}^m$ .

However, the first option is incompatible with (iv) because it is impossible for  $(G/H, g_{a(s),b(s)})$  to converge in the Hausdorff sense to a metric space homeomorphic to  $L_{m,2k}$ . Indeed, any change in topology must arise from sending  $a$  or  $b$  to zero, which necessarily results in a dimension drop. Hence the second option holds and  $X \cong \mathbb{C}\mathbb{P}^m$ .

For  $(G/H, g_{a(s),b(s)})$  to converge in the Hausdorff sense to a metric on  $\mathbb{C}\mathbb{P}^m$ , we need  $a(s) \rightarrow 0$  and  $b(s) \rightarrow b(0) > 0$  as  $s \rightarrow 0$ . For the metric to close up smoothly at  $X$ , we adapt the discussion in [36, §3] to obtain  $a'(0) = k$  and  $b'(0) = 0$  as necessary conditions. As  $f$  is a function of the radial coordinate  $s$  on fibers, we also see by restricting to a single fiber that smoothness requires  $f'(0) = 0$ . Thus, the system (1.6)–(1.8) holds with boundary conditions given by (1.9). If  $(M, g, f)$  is an asymptotically  $L_{m,k}$ -cylindrical steady GRS, then this and (5.12) imply that  $(M, g, f) \in \mathcal{A}_{m,k}^s$  (as defined in §1.3). If  $(M, g, f)$  is an asymptotically  $(L_{m,k}, g_{\alpha,\beta})$ -conical expanding GRS, then this and (5.13) imply that  $(M, g, f) \in \mathcal{A}_{m,k}^e(\alpha, \beta)$ .  $\square$

## REFERENCES

- [1] M. M. Alexandrino and R. G. Bettiol, *Lie groups and geometric aspects of isometric actions*, Springer Cham, 2015.
- [2] A. Appleton, *A family of non-collapsed steady Ricci solitons in even dimensions greater or equal to four*, Preprint, [arXiv:1708.00161](https://arxiv.org/abs/1708.00161) (2017).
- [3] R. H. Bamler, B. Chow, Y. Deng, Z. Ma, and Y. Zhang, *Four-dimensional steady gradient Ricci solitons with 3-cylindrical tangent flows at infinity*, *Adv. Math.* **401** (2022), no. 108285.
- [4] S. Brendle, *Rotational symmetry of self-similar solutions to the Ricci flow*, *Invent. Math.* **194** (2013), 731–764.
- [5] ———, *Rotational symmetry of Ricci solitons in higher dimensions*, *J. Differential Geom.* **97** (2014), no. 2, 191–214.
- [6] ———, *Ancient solutions to the Ricci flow in dimension 3*, *Acta Math.* **225** (2020), 1–102.
- [7] S. Brendle, P. Daskalopoulos, and N. Sesum, *Uniqueness of compact ancient solutions to three-dimensional Ricci flow*, *Invent. Math.* **226** (2021), 579–651.
- [8] S. Brendle and K. Naff, *Rotational symmetry of ancient solutions to the Ricci flow in higher dimensions*, *Geom. Topol.* **27** (2023), no. 1, 153–226.
- [9] R. L. Bryant, *Ricci flow solitons in dimension three with  $SO(3)$ -symmetries*, Preprint, <https://services.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf> (2005).
- [10] H.-D. Cao, *Existence of gradient Kähler-Ricci solitons*, *Elliptic and Parabolic Methods in Geometry* (Minneapolis, MN, 1994), A K Peters, 1996.
- [11] H.-D. Cao and M. Zhu, *On second variation of Perelman’s Ricci shrinker entropy*, *Math. Ann.* **353** (2012), 747–763.
- [12] O. Chodosh, *Expanding Ricci solitons asymptotic to cones*, *Calc. Var. Partial Differential Equations* **51** (2014), 1–15.
- [13] O. Chodosh, K. Choi, C. Mantoulidis, and F. Schulze, *Mean curvature flow with generic initial data*, *Invent. Math.* **237** (2024), 121–220.
- [14] O. Chodosh and F. T.-H. Fong, *Rotational symmetry of conical Kähler-Ricci solitons*, *Math. Ann.* **364** (2016), 777–792.
- [15] K. Choi, R. Haslhofer, O. Hershkovits, and B. White, *Ancient asymptotically cylindrical flows and applications*, *Invent. Math.* **229** (2022), 134–241.
- [16] B. Chow, *Ricci solitons in low dimensions*, *Graduate Studies in Mathematics*, vol. 235, American Mathematical Society, 2023.
- [17] B. Chow, Y. Deng, and Z. Ma, *On four-dimensional steady gradient Ricci solitons that dimension reduce*, *Adv. Math.* **403** (2022), no. 108367.
- [18] B. Chow, P. Lu, and L. Ni, *Hamilton’s Ricci flow*, *Graduate Studies in Mathematics*, vol. 77, American Mathematical Society, 2006.
- [19] T. H. Colding and W. P. Minicozzi II, *Generic mean curvature flow I; generic singularities*, *Ann. of Math.* **175** (2012), 755–833.
- [20] R. J. Conlon, A. Deruelle, and S. Sun, *Classification results for expanding and shrinking gradient Kähler-Ricci solitons*, *Geom. Topol.* **28** (2024), no. 1, 267–351.

- [21] G. de Rham, *Complexes à automorphismes et homéomorphie différentiable*, Ann. Inst. Fourier **2** (1950), 51–67.
- [22] Y. Deng, *Rigidity of positively curved steady gradient Ricci solitons on orbifolds*, Preprint, [arXiv:2504.14525](https://arxiv.org/abs/2504.14525) (2025).
- [23] Y. Deng and X. Zhu, *Classification of gradient steady Ricci solitons with linear curvature decay*, Sci. China Math. **63** (2020), 135–154.
- [24] A. Deruelle, *Stability of non compact steady and expanding gradient Ricci solitons*, Calc. Var. Partial Differential Equations **54** (2015), 2367–2405.
- [25] P. Donovan, *Cohomogeneity one 4-dimensional gradient Ricci solitons*, J. Geom. Anal. **35** (2025), no. 401.
- [26] M. Feldman, T. Ilmanen, and D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons*, J. Differential Geom. **65** (2003), 169–209.
- [27] O. Hershkovits, *Translators asymptotic to cylinders*, J. Reine Angew. Math. **766** (2020), 61–71.
- [28] C. A. Hoelscher, *Classification of cohomogeneity one manifolds in low dimensions*, Pacific J. Math. **246** (2010), no. 1, 129–185.
- [29] S. Kobayashi, *Transformation groups in differential geometry*, Classics in Mathematics, Springer Berlin, Heidelberg, 1995.
- [30] M. B. Law, *An analytical characterization of Eguchi–Hanson space and its higher-dimensional analogs*, in preparation.
- [31] P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math. **65** (1957), no. 3, 447–455.
- [32] O. Munteanu and J. Wang, *Smooth metric measure spaces with non-negative curvature*, Comm. Anal. Geom. **19** (2011), no. 3, 451–486.
- [33] K. Naff and T. Ozuch, *Linear stability of the blowdown Ricci shrinker in 4D*, Preprint, [arXiv:2509.05470](https://arxiv.org/abs/2509.05470) (2025).
- [34] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, [arXiv:math/0211159](https://arxiv.org/abs/math/0211159) (2002).
- [35] F. Torralbo and F. Urbano, *Index of compact minimal submanifolds of the Berger spheres*, Calc. Var. Partial Differential Equations **61** (2022), no. 104.
- [36] L. Verdiani and W. Ziller, *Smoothness conditions in cohomogeneity one manifolds*, Transform. Groups **27** (2022), 311–342.
- [37] Z. Zhao and X. Zhu, *Rigidity of the Bryant Ricci soliton*, Preprint, [arXiv:2212.02889](https://arxiv.org/abs/2212.02889) (2022).
- [38] W. Ziller, *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259** (1982), 351–358.

MIT, DEPARTMENT OF MATHEMATICS, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139, USA.  
 Email address: [mikelaw@mit.edu](mailto:mikelaw@mit.edu)