The No Wandering Domains theorem

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1 Introduction

Let f be a map from the Riemann sphere $\hat{\mathbb{C}}$ to itself. In complex dynamics, one investigates the behaviour of the family of iterates $\{f^{\circ n} : n \in \mathbb{N}\}$, where $f^{\circ n} = f \circ f \circ \ldots \circ f$ is the n-fold composition of f. The study of iterated maps gives rise to arguably some of the most beautiful mathematics through its intimate links with fractal geometry. In this report, we will prove the *No Wandering Domains Theorem*, which was first proved by Sullivan [1]. This says that when f is rational with degree ≥ 2 (degree being the larger polynomial degree between the numerator and denominator), a sufficiently nice connected open set eventually returns to itself under the family of iterates of f.

Theorem 1.1 (No Wandering Domains)**.** *A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with degree $d \geq 2$ has no wandering domains.*

In Section 2, we briefly run through the definitions required to understand the statement of Theorem 1.1. The precise definition of a wandering domain is given, of course. In Section 3, we introduce quasiconformal maps and Beltrami forms, and state the celebrated Measurable Riemann Mapping Theorem which is central to the proof of Theorem 1.1. We proceed with the proof in Section 4. Our treatment follows [2] and [3], although we elaborate more at times. [4] was also consulted as a reference.

2 Fatou and Julia sets

We give just enough background on Fatou and Julia sets to understand the statement of Theorem 1.1.

Definition 2.1. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a nonconstant holomorphic map. A* ***periodic point*** *of f with period n is a point z such that $f(z), (f \circ f)(z), \ldots, f^{\circ(n-1)}(z) \neq z$, but $f^{\circ n}(z) = z$. The* ***multiplier*** *of a periodic point z with period n is the number $(f^{\circ n})'(z)$.*

Definition 2.2. *The* ***Julia set*** *of f, denoted $J(f)$, is the closure of the set of periodic points of f with multiplier having modulus > 1. The* ***Fatou set*** *of f is $F(f) = \hat{\mathbb{C}} \setminus J(f)$.*

Note that this implies $F(f)$ is open in $\hat{\mathbb{C}}$. We remark that these are one of many equivalent definitions of the Fatou and Julia sets of f, and we have chosen this one to keep things simple.

If z belongs to the interior of $J(f)$ with period n, then from the definition we see that $|f^{\circ n}(w) - f^{\circ n}(z)| = |f^{\circ n}(w) - z| > |w - z|$ for nearby points w. Thus, points near z are *repelled* from z as we iterate f, leading to chaotic behaviour near z. The converse holds true in general: points in $F(f)$ are well-behaved under the iterates of f. Now comes the obligatory picture of $J(f)$ and $F(f)$...

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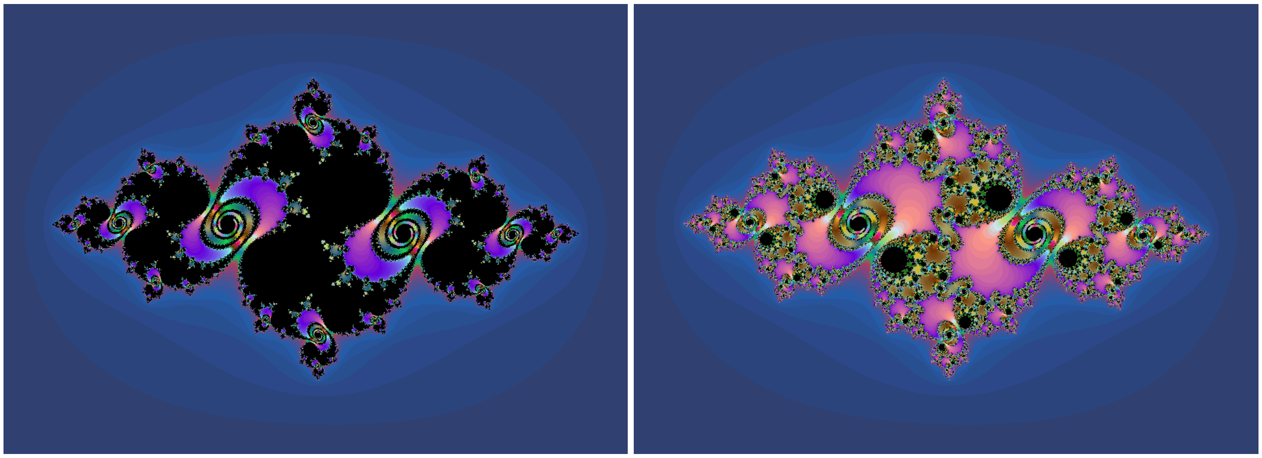

Figure 1: Fatou and Julia sets for $f(z) = z^2 + c$ where $c = -0.755 + 0.05i$ (left) and $c = -0.755 + 0.0575i$ (right), in a neighbourhood of the origin. The black region is $J(f)$, and the colours represent divergence rate of points in $F(f)$ to $\infty \in \hat{\mathbb{C}}$. Source: `https://mathlair.allfunandgames.ca/julia2.php`.

Here are two key properties of $F(f)$ and $J(f)$, which are easier proved using alternative definitions.

Lemma 2.3. *The sets $F(f)$ and $J(f)$ are* invariant *under iterations of f. That is, a point $z \in \hat{\mathbb{C}}$ belongs to $F(f)$ (resp. $J(f)$) if and only if $f(z)$ belongs to $F(f)$ (resp. $J(f)$).*

Lemma 2.4. *For a rational map of degree ≥ 2, the Julia set is nonempty.*

Another reason why the Fatou set is considered 'nice' is due to the following.

Lemma 2.5. *For every* ***Fatou component*** *U (i.e. connected component of the Fatou set) of a rational map f, $f(U)$ is another Fatou component, in fact the whole component.*

Proof. Since f is holomorphic (thus an open map) and U is connected and open, $f(U)$ is also connected and open. In particular $f(U)$ is open in $F(f)$ since $f(U) \subseteq F(f)$. We also have $U \cap \partial U = \emptyset$, so $U = \bar{U} \cap F(f)$ and

$$f(U) = f(\bar{U}) \cap f(F(f)) = f(\bar{U}) \cap F(f), \tag{1}$$

using the invariance of $F(f)$ (Lemma 2.3). But $\bar{U}$ is compact, being a closed subspace of $\hat{\mathbb{C}}$. So $f(\bar{U})$ is also compact, hence closed since $\hat{\mathbb{C}}$ is Hausdorff. By (1), $f(U)$ is closed in $F(f)$, in addition to being open and connected. Hence it is a whole connected component of $F(f)$. □

Definition 2.6. *Let U be a Fatou component of f. If there exists $n < \infty$ such that $f^{\circ n}(U) = U$, then U is called* ***eventually periodic****. Otherwise, $f^{\circ n}(U) \cap U = \emptyset$ for all n (by Lemma 2.5), and U is called a* ***wandering domain****.*

Now we can make sense of Theorem 1.1, which in other words says that every Fatou component of a rational map of degree ≥ 2 is eventually periodic. This is by no means obvious, and perhaps surprising: such a map only has a handful of fixed points if any, but *every* Fatou component is a 'fixed component', at least under some iterate of the map. Theorem 1.1 applies for both maps displayed in Figure 1.

3 Quasiconformal maps and Beltrami forms

The proof of Theorem 1.1 heavily relies on Beltrami forms and the Measurable Riemann Mapping Theorem, which both arise in the theory of quasiconformal maps. We introduce these ideas with some intuition.

Definition 3.1. *A* ***conformal structure*** *on a smooth manifold M is an equivalence class of Riemannian metrics on M, where two metrics g and h are* ***conformally equivalent*** *if $g = \lambda h$ for some smooth positive function $\lambda : M \to \mathbb{R}$.*

A conformal structure is an angle-measuring device at each point on a manifold. A metric would have also done the job, but the equivalence relation of conformality allows us to forget about distances, areas, etc.

Definition 3.2. *Let M and N be smooth manifolds with conformal structures $[g]$ and $[h]$ respectively, where g and h are representative metrics. A map $f : M \to N$ is* ***conformal*** *if the pullback metric f^*h on M is conformally equivalent to g wherever defined. If such a map exists and is a diffeomorphism, we call M and N* ***conformally isomorphic****.*

We restrict our attention to surfaces, i.e. real dimension 2. The following fact is imperative to mention.

Theorem 3.3. *On a surface, there is a one-to-one correspondence between conformal and complex structures.*

Before generalising, consider when the surface is an open set $U \subseteq \mathbb{C}$. How do we specify a conformal structure on U? By definition, it suffices to provide a metric g on U, as its conformal class will be our conformal structure. Actually, we only need to specify the unit ball of the norm induced by g (in each tangent space), as then g is recovered by the polarisation identity.

Taking this further: for each $p \in U$, the unit ball of $g|_{T_pU}$ is an ellipse in $T_pU \cong \mathbb{C}$. This has locus $|az + b\bar{z}| = 1$ for some $a, b \in \mathbb{C}$, or $|z + \mu\bar{z}| = c$ for some $\mu, c \in \mathbb{C}$.[1] Allowing $p \in U$ to vary, μ and c become smooth functions of p. Now let h be a metric which is conformally equivalent to g. Then $h = \lambda g$ for a smooth $\lambda : U \to \mathbb{R}^+$, so the unit circle of h in T_pU is $|z + \mu(p)\bar{z}| = \lambda(p)c(p)$, with $\lambda(p)c(p)$ smooth in p. From this, we see that the function μ is an invariant of the conformal class. Conversely, if two metrics generate unit balls $|z + \mu(p)\bar{z}| = c_1(p)$ and $|z + \mu(p)\bar{z}| = c_2(p)$ at T_pU, then these ellipses are concentric, so the metrics differ only by a positive multiple at each point.[2] Thus, they are conformally equivalent.

The upshot is as follows. To give a conformal structure on U, it is enough to specify a smooth function $\mu : U \to \mathbb{C}$. This μ describes the field of concentric ellipses at the tangent spaces, i.e. the unit circles of its representing metrics. In particular it contains information about their orientation and eccentricity, e.g. the eccentricity of the ellipses at T_pU is $\frac{1+|\mu(p)|}{1-|\mu(p)|}$ if $|\mu(p)| < 1$, and $\frac{1-|\mu(p)|}{1+|\mu(p)|}$ if $|\mu(p)| \geq 1$.

Given a domain $U \subseteq \mathbb{C}$ and a smooth function $\mu : U \to \mathbb{C}$, denote by U_μ the surface U with conformal structure described by μ. The next theorem tells us when a function $f : U_\mu \to V_\nu$ between open sets of $\mathbb{C}$ is conformal. A special case is when $V_\nu = \mathbb{C}_0$, the complex plane with the standard conformal structure. Here 0 is the zero function.

[1] If $a = 0$, just take $\mu = 0$.

[2] Positivity comes from the positive-definiteness of Riemannian metrics.

Theorem 3.4. *A smooth function $f : U_\mu \to V_\nu$ is conformal if $\partial f/\partial z \neq 0$ and*

$$\frac{\partial f}{\partial \bar{z}} + \nu(f(z))\frac{\partial \bar{f}}{\partial \bar{z}} = \mu(z)\left(\frac{\partial f}{\partial z} + \nu(f(z))\frac{\partial \bar{f}}{\partial z}\right), \quad \forall z \in U. \tag{2}$$

If $V_\nu = \mathbb{C}_0$, (2) *reduces to the* ***Beltrami equation****:*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}, \quad \forall z \in U. \tag{3}$$

Proof. We prove (3); the idea is the same for (2). So $\nu = 0$, and the ellipses $|w + \nu\bar{w}| = |w| = \text{const}$ are circles. By our previous discussion and the definition of a conformal map, we want to show that circles in $T_{f(z)}V$ are pulled back to ellipses $|w + \mu(z)\bar{w}| = \text{const}$ in T_zU. Equivalently, it suffices to show that for all $z \in U$ the derivative df takes the ellipse of tangent vectors $|w + \mu(z)\bar{w}| = \text{const}$ in T_zU to a circle $|df(w)| = \text{const}$ in $T_{f(z)}\mathbb{C} \cong \mathbb{C}$. This indeed holds, since for $z \in U$ and $w \in T_zU$ we have

$$|df(w)|^2 = \left|w\frac{\partial f}{\partial z} + \bar{w}\frac{\partial f}{\partial \bar{z}}\right|^2 = \left|\frac{\partial f}{\partial z}\right|^2 \left|w + \frac{\partial f/\partial \bar{z}}{\partial f/\partial z}\bar{w}\right|^2 = \left|\frac{\partial f}{\partial z}\right|^2 |w + \mu(z)\bar{w}|^2 = \text{const}.$$

□

Definition 3.5. *If $f : U_\mu \to \mathbb{C}_0$ satisfies* (3)*, we call the same map $f : U_0 \to \mathbb{C}_0$ μ-**quasiconformal** when the domain has the standard conformal structure, and we say f **satisfies the Beltrami equation** for μ.*

For open sets in $\mathbb{C}$, we see that a single function μ defines a conformal structure. Can we proceed similarly for a Riemann surface X, i.e. define a new conformal structure on X by assigning functions $\mu : \varphi(U) \to \mathbb{C}$ to each chart $(U \subseteq X, \varphi : U \to \mathbb{C})$, and pulling back the conformal structures on $\varphi(U)$ induced by μ to get local conformal structures on U? Suppose we have assigned such functions; let us derive the constraints. Let z, w be local coordinates on overlapping patches of X, and suppose the conformal structures in the charts are given by $\mu(z)$ and $\nu(w)$ respectively. To ensure the result is still a Riemann surface, the transition map $z \mapsto w(z)$ must be holomorphic. So it must be μ-ν-conformal, and hence by (2) we can impose the requirement that

$$\nu(w(z)) = \mu(z)\frac{\partial w}{\partial z}\frac{\partial \bar{z}}{\partial \bar{w}}, \tag{4}$$

since $\partial w/\partial \bar{z} = 0 = \partial \bar{w}/\partial z$. This condition on the functions μ, ν therefore ensures they define a consistent conformal structure on X (hence complex structure by Theorem 3.3). The next definition is now natural.

Definition 3.6. *A **Beltrami form** on a surface X is a collection of smooth functions $\mathcal{B}$ each defined on an open set of X, such that the transition maps between $\mu, \nu \in \mathcal{B}$ satisfy* (4)*.*

An equivalent, more convenient definition is as follows. The equivalence can be seen by 'moving all the w's to the left-hand side' in (4).

Definition 3.7. *A **Beltrami form** on X is a tensor field over X locally represented by $\mu = \mu(z)\, d\bar{z} \otimes \frac{\partial}{\partial z}$ for a smooth function $\mu(z)$. Sometimes μ is referred to as a $(-1,1)$-tensor field by the way it transforms.*

In summary, a Beltrami form μ on X defines a new Riemann surface homeomorphic to X, but in general, conformally inequivalent. Denote the new Riemann surface by X_μ. We will focus only on the case $X = \hat{\mathbb{C}}$, in which case $\hat{\mathbb{C}}_\mu$ *is* actually conformally isomorphic to $\hat{\mathbb{C}}_0$ by the Uniformisation Theorem.[3]

Remark. *From here on, we allow objects to be merely measurable. A careful treatment is too cumbersome, but any subtleties will not cause issues in the present discussion anyway. In essence, we now allow things to be defined up to sets of zero measure, and everything from here on should be suffixed with 'a.e.' where appropriate. Previous definitions and theorems should also be replaced with their measurable equivalents.*

We now state the Measurable Riemann Mapping Theorem (MRMT), due to Alfohrs and Bers. It asserts the existence of a distinguished conformal map $\hat{\mathbb{C}}_\mu \to \hat{\mathbb{C}}_0$ subject only to mild conditions on μ. Even more, it says that these maps for nearby μ are related by a 'holomorphic variation':

Theorem 3.8 (MRMT)**.** *Let μ be a measurable Beltrami form on $\hat{\mathbb{C}}$ with* $\operatorname{ess\,sup}|\mu| < 1$.[4] *There is a unique L^1 conformal homeomorphism $f : \hat{\mathbb{C}}_\mu \to \hat{\mathbb{C}}_0$ satisfying the Beltrami equation $\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}$, and fixes $0, 1, \infty$.*[5] *Moreover, the map $f_t(z)$ obtained this way for the Beltrami form $t\mu$ (for small enough complex t) is holomorphic in t for fixed z.*

As above, we call this function $f : \hat{\mathbb{C}}_0 \to \hat{\mathbb{C}}_0$ **μ-quasiconformal** when the domain has the usual conformal structure on $\hat{\mathbb{C}}$. Note that by setting $t = 0$ in the second part of the theorem, f_0 is the identity map on $\hat{\mathbb{C}}$.[6]

4 Proof of the No Wandering Domains theorem

The proof of Theorem 1.1 is an unexpected yet intriguing application of Beltrami forms and the MRMT. In addition to this, the proof strategy seems rather arbitrary, making its execution all the more surprising.

We first give a sketch. For a Riemann surface X, denote by $M(X)$ the $\mathbb{C}$-vector space of essentially bounded Beltrami forms on X. Let $M(\hat{\mathbb{C}})^f \subseteq M(\hat{\mathbb{C}})$ be the subspace of *f-invariant* Beltrami forms on $\hat{\mathbb{C}}$, i.e. those $\hat{\mu} \in M(\hat{\mathbb{C}})$ such that $f : \hat{\mathbb{C}}_{\hat{\mu}} \to \hat{\mathbb{C}}_{\hat{\mu}}$ is conformal. According to (2) with $\mu = \nu = \hat{\mu}$ and noting that $\partial f/\partial \bar{z} = 0$ by holomorphicity, we have $\hat{\mu} \in M(\hat{\mathbb{C}})^f$ whenever

$$\hat{\mu}(f(z))\overline{f'(z)} = \hat{\mu}(z)f'(z) \quad \text{a.e.} \tag{5}$$

(We only need to care about one of the two charts on $\hat{\mathbb{C}}$ since the uncovered part has zero measure.) For a contradiction, suppose f has a wandering domain U. We will construct linear maps

$$M(U) \to M(\hat{\mathbb{C}})^f \to T_f\mathrm{Rat}_d, \tag{6}$$

where Rat_d is the set of rational maps of degree d on $\hat{\mathbb{C}}$. These have $2d + 2$ coefficients and are scale-invariant, so Rat_d is open in $\mathbb{CP}^{2d+1}$, making it a $(2d + 1)$-dimensional complex manifold.

[3] $\hat{\mathbb{C}}_0$ is S^2 with the conformal structure induced by the usual two charts, both with the usual conformal structure on $\mathbb{C}$.

[4] More precisely, if we pick one of the two charts on $\hat{\mathbb{C}}$ and write $\mu = \mu(z)\, d\bar{z} \otimes \frac{\partial}{\partial z}$ in that chart, then $\operatorname{ess\,sup}_z |\mu(z)| < 1$.

[5] These derivatives are L^1 distributional derivatives.

[6] See Homework 2, Problem 3a.

Elements of $T_f \text{Rat}_d$ are holomorphic sections of the bundle $f^* T\hat{\mathbb{C}}$ over $\hat{\mathbb{C}}$, i.e. the fiber over z is $T_{f(z)}\hat{\mathbb{C}}$ (this will not be proved).

We will show that (6) restricts to an injective map on an infinite-dimensional linear subspace V of $M(U)$, which yields a contradiction since the codomain $T_f \text{Rat}_d$ is finite-dimensional.

The following lemma helps simplify things. We omit the proof, which is a little technical.

Lemma 4.1 ([5]). *If U is a wandering domain of f, then for all sufficiently large n, $f^{\circ n}(U)$ is simply connected, and f maps $f^{\circ n}(U)$ homeomorphically onto $f^{\circ(n+1)}(U)$.*

Proof of Theorem 1.1. Assume f has a wandering domain $U \subseteq F(f)$. By Lemma 4.1, sufficiently high iterates of U under f are simply connected, so we may assume U is simply connected from the outset. First, we define an injective linear map $M(U) \to M(\hat{\mathbb{C}})^f$. For each $\mu \in M(U)$, define $\hat{\mu} \in M(\hat{\mathbb{C}})^f$ by 'distributing' μ over almost all of $\hat{\mathbb{C}}$ in the following way:

(i) Set $\hat{\mu} = \mu$ on U.

(ii) On each forward iterate $f^{\circ n}(U)$, recursively set $\hat{\mu}(z) = \hat{\mu}(w) f'(w)/\overline{f'(w)}$ where $w = f^{-1}(z) \in f^{\circ(n-1)}(U)$ (this is well-defined since f maps all large iterates of U bijectively by Lemma 4.1).

(iii) On each backward iterate $f^{\circ(-n)}(U) = (f^{\circ n})^{-1}(U)$, recursively set $\hat{\mu}(z) = \hat{\mu}(f(z))\overline{f'(z)}/f'(z)$.

(iv) Set $\hat{\mu} = 0$ elsewhere.

The assumption that U is wandering is crucial, because for $\hat{\mu}$ to be well-defined the iterates $f^{\circ n}(U)$ must be disjoint. Moreover, this defines $\hat{\mu}$ on all of $\hat{\mathbb{C}}$ except at the points z where some iterate of z is a critical point of f, as this causes division by zero in (ii) and (iii).[7] These points form a countable set whose measure is therefore null, so we leave $\hat{\mu}$ undefined there; we only aim to construct a *measurable* Beltrami form.

Because μ is essentially bounded, so is $\hat{\mu}$ (as $|\overline{f'}/f| = |f/\overline{f'}| = 1$ for (ii) and (iii)). Moreover, $\hat{\mu}$ is manifestly f-invariant since it satisfies (5) by construction. Hence $\hat{\mu} \in M(\hat{\mathbb{C}})^f$. By (i), this assignment of $\hat{\mu}$ to each $\mu \in M(U)$ is injective. It is also linear, as is easily seen from (i)-(iv).

Now that we have an injective linear map $M(U) \to M(\hat{\mathbb{C}})^f$, we complete (6) by constructing a linear map $M(\hat{\mathbb{C}})^f \to T_f \text{Rat}_d$. Given $\hat{\mu} \in M(\hat{\mathbb{C}})^f$, essential boundedness implies $\operatorname{ess\,sup} |t\hat{\mu}| < 1$ for small enough t. By Theorem 3.8, there is a one-parameter family of conformal homeomorphisms $\phi_t : \hat{\mathbb{C}}_{t\hat{\mu}} \to \hat{\mathbb{C}}_0$ which fix $0, 1, \infty$, satisfy the Beltrami equation for $t\hat{\mu}$, and depend holomorphically on t for fixed $z \in \hat{\mathbb{C}}$. The criterion (5) for f-invariance is linear in $\hat{\mu}$, so the f-invariance of $\hat{\mu}$ implies f-invariance of $t\hat{\mu}$. Thus,

$$\phi_t \circ f \circ \phi_t^{-1} : \hat{\mathbb{C}}_0 \xrightarrow{\phi_t^{-1}} \hat{\mathbb{C}}_{t\hat{\mu}} \xrightarrow{f} \hat{\mathbb{C}}_{t\hat{\mu}} \xrightarrow{\phi_t} \hat{\mathbb{C}}_0$$

is a composition of conformal maps, hence is conformal. So $\phi_t \circ f \circ \phi_t^{-1}$ is a rational map on $\hat{\mathbb{C}}$ with the same degree as f for each t, and the dependence on t is holomorphic. Let $w_{\hat{\mu}}$ be the first variation of this family:

$$w_{\hat{\mu}}(z) = \frac{\partial}{\partial t}\Big|_{t=0} (\phi_t \circ f \circ \phi_t^{-1})(z), \tag{7}$$

[7] Specifically, for $z \in f^{\circ(-n)}(U)$ (resp. $f^{\circ n}(U)$), $\hat{\mu}(z)$ is undefined if some forward (backward) iterate of z is a critical point of f.

which we identify with an element of $T_f\mathrm{Rat}_d$. (We can treat the expression on the right as a complex-valued function, in which case it should really be multiplied by $\frac{\partial}{\partial z}$.) Declare the second map in (6) to be

$$M(\hat{\mathbb{C}})^f \to T_f\mathrm{Rat}_d, \quad \hat{\mu} \mapsto w_{\hat{\mu}}.$$

To see that this is linear, consider the vector field on $\hat{\mathbb{C}}$ defined by

$$v_{\hat{\mu}} \in \Gamma(T\hat{\mathbb{C}}), \quad v_{\hat{\mu}}(z) = \frac{\partial}{\partial t}\Big|_{t=0} \phi_t(z). \tag{8}$$

(Same comment about treating the right-hand side as a function.) Then:

- $\underline{w_{\hat{\mu}} \text{ depends linearly on } v_{\hat{\mu}}}$: by writing $\phi_t(z) = \phi(t, z)$ and recalling that $\phi(0, z) = \phi^{-1}(0, z) = z$ (see the end of Section 3) we compute by the chain rule

$$\begin{aligned} w_{\hat{\mu}}(z) &= \frac{\partial}{\partial t}\Big|_{t=0} (\phi_t \circ f \circ \phi_t^{-1})(z) = \frac{\partial}{\partial t}\Big|_{t=0} \phi(t, f(\phi^{-1}(t, z))) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \phi(t, f(\phi^{-1}(0, z))) + \frac{\partial}{\partial t}\Big|_{t=0} \phi(0, f(\phi^{-1}(t, z))) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \phi(t, f(z)) + \frac{\partial}{\partial t}\Big|_{t=0} f(\phi^{-1}(t, z)) \\ &= v_{\hat{\mu}}(f(z)) + f'(\phi^{-1}(0, z)) \cdot \frac{\partial}{\partial t}\Big|_{t=0} \phi_t^{-1}(z) \\ &= v_{\hat{\mu}}(f(z)) + f'(z) \cdot \frac{\partial}{\partial t}\Big|_{t=0} \phi_{-t}(z) \\ &= v_{\hat{\mu}}(f(z)) - f'(z) v_{\hat{\mu}}(z). \end{aligned} \tag{9}$$

- $\underline{v_{\hat{\mu}} \text{ depends linearly on } \hat{\mu}}$: write $v_{\hat{\mu}}(z) = \left(\frac{\partial}{\partial t}\Big|_{t=0} \phi_t(z)\right) \frac{\partial}{\partial z}$, now treating the expression in parentheses as a $\mathbb{C}$-valued function in z. Defining $\bar{\partial}$ on $\frac{\partial}{\partial z}$-vector fields by $\bar{\partial}\left(g \frac{\partial}{\partial z}\right) = (\bar{\partial} g)\, d\bar{z} \otimes \frac{\partial}{\partial z}$, we compute

$$\begin{aligned} \bar{\partial} v_{\hat{\mu}} &= \frac{\partial}{\partial \bar{z}} \left(\frac{\partial}{\partial t}\Big|_{t=0} \phi_t(z) \right) d\bar{z} \otimes \frac{\partial}{\partial z} = \frac{\partial}{\partial t}\Big|_{t=0} \left(\frac{\partial}{\partial \bar{z}} \phi_t(z) \right) d\bar{z} \otimes \frac{\partial}{\partial z} \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \left(t\hat{\mu}(z) \frac{\partial}{\partial z} \phi_t(z) \right) d\bar{z} \otimes \frac{\partial}{\partial z} \qquad (\phi_t \text{ satisfies Beltrami eqn. for } t\hat{\mu}) \\ &= \left[\left(\hat{\mu}(z) \frac{\partial}{\partial z} \phi_t(z) \right) \Big|_{t=0} + \left(t\hat{\mu}(z) \frac{\partial}{\partial t} \frac{\partial}{\partial z} \phi_t(z) \right) \Big|_{t=0} \right] d\bar{z} \otimes \frac{\partial}{\partial z} \\ &= \left(\hat{\mu}(z) \frac{\partial}{\partial z} z \right) d\bar{z} \otimes \frac{\partial}{\partial z} = \hat{\mu}(z)\, d\bar{z} \otimes \frac{\partial}{\partial z} = \hat{\mu}. \end{aligned} \tag{10}$$

  In fact, $v_{\hat{\mu}}$ is the unique solution to $\bar{\partial} v = \mu$ vanishing at $0, 1$ and ∞ by an infinitesimal version of the MRMT [2]. (The vanishing is because ϕ_t fixes $0, 1, \infty$ for all t.) Then for all $\hat{\mu}, \hat{\lambda} \in M(\hat{\mathbb{C}})^f$ and $c \in \mathbb{C}$ we have $c\hat{\mu} + \hat{\lambda} = c\bar{\partial} v_{\hat{\mu}} + \bar{\partial} v_{\hat{\lambda}} = \bar{\partial}(c v_{\hat{\mu}} + v_{\hat{\lambda}})$, hence $v_{c\hat{\mu}+\hat{\lambda}} = c v_{\hat{\mu}} + v_{\hat{\lambda}}$ by uniqueness of the solution. This proves linearity of $v_{\hat{\mu}}$ with respect to $\hat{\mu} \in M(\mathbb{C})^f$.

We now have linear maps as in (6), so the final step is to exhibit an infinite-dimensional subspace V of $M(U)$ on which (6) is injective, thereby forcing a contradiction. Take V as follows.

Lemma 4.2. *There is an infinite-dimensional subspace V of $M(U)$ consisting of compactly supported Beltrami forms μ such that if $\bar{\partial} v = \mu$ for some continuous vector field with $v|_{\partial U} = 0$, then $\mu = 0$.*

Before proving this, let us accept its truthhood and finish proving Theorem 1.1. Consider the composition

$$V \hookrightarrow M(U) \to M(\hat{\mathbb{C}})^f \to T_f \mathrm{Rat}_d, \tag{11}$$

which is the inclusion $V \hookrightarrow M(U)$ composed with (6). Since (6) is linear, so is (11). We show it is also injective. Suppose $\mu \in V \subseteq M(U)$ maps to zero under this composition. That is, $w_{\hat{\mu}} = 0$ where $\hat{\mu}$ is the extension of μ provided by the map $M(U) \to M(\hat{\mathbb{C}})^f$, and $w_{\hat{\mu}}$ is defined in (7). By (9), we have

$$v_{\hat{\mu}}(f(z)) = f'(z) v_{\hat{\mu}}(z). \tag{12}$$

Let z be a point in the Julia set $J(f)$, with period n and multiplier λ where $|\lambda| > 1$; see Definition 2.2 and Lemma 2.4. We get n equations generated by (12):

$$v_{\hat{\mu}}((f^{\circ(j+1)}(z)) = f'(f^{\circ j}(z)) v_{\hat{\mu}}(f^{\circ j}(z)), \quad j = 0, \dots, n-1.$$

Multiplying these (treating $v_{\hat{\mu}}$ as a complex-valued function) and using $v_{\hat{\mu}}(f^{\circ n}(z)) = v_{\hat{\mu}}(z)$,

$$\prod_{j=1}^{n} v_{\hat{\mu}}(f^{\circ j}(z)) = \underbrace{f'(f^{\circ(n-1)}(z)) \cdots f'(f(z)) f'(z)}_{=(f^{\circ n})'(z) = \lambda} \prod_{j=0}^{n-1} v_{\hat{\mu}}(f^{\circ j}(z))$$

$$\Rightarrow \quad (\lambda - 1) \prod_{j=0}^{n-1} v_{\hat{\mu}}(f^{\circ j}(z)) = 0.$$

As $|\lambda| > 1$, at least one of the $v_{\hat{\mu}}(f^{\circ j}(z))$ is zero. Using this fact, iterating (12) and applying the periodicity $v_{\hat{\mu}}(f^{\circ n}(z)) = v_{\hat{\mu}}(z)$, we see that $v_{\hat{\mu}}(z) = 0$. By the arbitrariness of $z \in J(f)$, it follows that $v_{\hat{\mu}} = 0$ on $J(f)$. In particular, $v_{\hat{\mu}}|_{\partial U} = 0$ since $\partial U \cap F(f) = \emptyset \implies \partial U \subseteq J(f)$, recalling that $F(f)$ is open. But also $\bar{\partial} v_{\hat{\mu}} = \hat{\mu}$ by (10), and Lemma 4.2 now tells us that $\hat{\mu} = 0$. Obviously $\mu = 0$ follows, as $\mu = \hat{\mu}|_U$. Hence (11) is injective, yielding the desired contradiction since V is infinite-dimensional while $T_f \mathrm{Rat}_d$ is finite-dimensional. □

Theorem 1.1 is thus proved. It remains to prove Lemma 4.2.

Proof of Lemma 4.2. Note that U is not conformal to $\hat{\mathbb{C}}$ nor $\mathbb{C}$, otherwise U *is* $\hat{\mathbb{C}}$ (respectively $\hat{\mathbb{C}} \setminus \{1 \text{ point}\}$) and it follows that U cannot be wandering. We assumed U is simply connected, so by the (usual) Riemann mapping theorem, a conformal isomorphism $\psi : U \to \mathbb{D}$ exists and induces a vector space isomorphism $M(U) \cong M(\mathbb{D})$. Hence to prove the lemma it suffices to prove the analogous statement in $\mathbb{D}$, as everything can then be pulled back along ψ to arrive at the result for U (see [2] for the exact details on that part).

We proceed by explicit constructing $V \subseteq M(\mathbb{D})$. For $k \in \mathbb{Z}_{\geq 0}$, define compactly supported measurable Beltrami forms $\mu_k \in M(\mathbb{D})$ by

$$\mu_k(z) = \begin{cases} \bar{z}^k d\bar{z} \otimes \frac{\partial}{\partial z} & \text{if } |z| \leq 1/2 \\ 0 & \text{if } |z| > 1/2. \end{cases}$$

Clearly the μ_k are $\mathbb{C}$-linearly independent, hence span an infinite dimensional subspace V of $M(\mathbb{D})$. The equation $\bar{\partial} v_k = \mu_k$ is satisfied for continuous vector fields v_k on $\mathbb{D}$ defined by

$$v_k(z) = \begin{cases} \frac{1}{k+1}\bar{z}^{k+1}\frac{\partial}{\partial z} & \text{if } |z| \leq 1/2 \\ \frac{1}{k+1}(4z)^{-(k+1)}\frac{\partial}{\partial z} & \text{if } |z| > 1/2. \end{cases} \tag{13}$$

In fact, these are unique solutions up to addition of holomorphic functions. Now let $\mu = \sum_{k=0}^{\infty} \lambda_k \mu_k \in V$ for some $\lambda_k \in \mathbb{C}$, and suppose $\bar{\partial} v = \mu$ has a solution with v continuous and $v|_{\partial \mathbb{D}} = 0$. To complete the proof we need to show that $\mu = 0$. Note that v is a $\mathbb{C}$-linear combination of the v_k's plus a holomorphic function, so v is holomorphic on $|z| > 1/2$ by (13). Because $v = 0$ on $|z| = 1$, we then have $v = 0$ on $1/2 \leq |z| \leq 1$ by the maximum modulus principle and the continuity of v.

For a contradiction, suppose $\mu \neq 0$. Then the λ_k's are not all zero. Let $w = \sum_{k=0}^{\infty} \lambda_k v_k$ be another vector field; we have $\bar{\partial} w = \sum_{k=0}^{\infty} \lambda_k \mu_k = \mu$. Then $\bar{\partial}(w - v) = 0$, so $w - v$ is holomorphic throughout $\mathbb{D}$. Since $v = 0$ on $1/2 \leq |z| \leq 1$, it follows that $w - v$ coincides with w there. Because w is a polynomial in z^{-1} there (as the λ_k's are not all zero), $w - v$ must be that same polynomial throughout $\mathbb{D}$. But this polynomial has no constant term (by the definition (13) and the fact that $k \geq 0$ in the definition of w), so $w - v$ has a singularity at zero. This contradicts the fact that $w - v$ is holomorphic in $\mathbb{D}$. Hence $\mu = 0$, and so V satisfies the conditions of the lemma in the unit disc case. □

References

[1] D. Sullivan (1985). *Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains*, Annals of Mathematics **122**, no. 3, 401–18.

[2] C. McMullen (2020). *Riemann Surfaces, Dynamics and Geometry*, `http://people.math.harvard.edu/~ctm/papers/home/text/class/notes/rs/course.pdf`

[3] O.A. Camarena (2016). *Iteration of Rational Functions*, `https://www.matem.unam.mx/~omar/no-wandering-domains.pdf`

[4] A. F. Beardon (1991). *Iteration of Rational Functions: Complex Analytic Dynamical Systems*. Graduate Texts in Mathematics, No. 132. Springer, New York.

[5] I. N. Baker (1984). *Wandering Domains in the Iteration of Entire Functions*. Proc. London Math. Soc. (3) **49**, 563-576.