Uniqueness of Tangent Flows in Mean Curvature Flow

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October 2021

A thesis submitted for the degree of
Bachelor of Mathematical Sciences (Honours)
at the Australian National University

Abstract

Tangent flows are limits obtained by rescaling solutions of mean curvature flow as they approach singularities. They are concrete geometric models of singularities, and are key to understanding the kinds of singularities that can occur. However, tangent flows are inherently defined as limits in a subconvergent sense, and this leaves open the possibility that two tangent flows associated to a given singularity look wildly different. The question of whether tangent flows are unique remains a major open problem, and an affirmative answer has striking implications for the singular set.

Uniqueness of tangent flows has long been known to hold for convex mean curvature flows in all dimensions and for all mean curvature flows in the plane (i.e. for the curve shortening flow). On the other hand, uniqueness under weaker assumptions has only recently shown promising signs of progress. This thesis details some of this recent work. The central result we present is that uniqueness of tangent flows holds too for mean convex mean curvature flows in all dimensions. To prove this, we will explain the work of Schulze and Colding–Minicozzi on the uniqueness of compact and cylindrical tangent flows respectively. We assume no prior familiarity with the mean curvature flow, so this thesis doubly functions as an entry point to this fascinating area of mathematics.

Declaration

The work in this thesis is my own except where otherwise stated.

Michael Law

Acknowledgements

I am deeply indebted to my supervisor, Ben Andrews, for his assistance and encouragement through every step of this project. Your expertise and patience is why I approach this piece of mathematics with confidence today – it's hard to believe this is the very same stuff that I felt so overwhelmed by just a few months ago. Thank you also for having the trust in me to steer this project in directions of my own choosing. Without your guidance, this thesis would be in a complete shambles.

I would like to thank Jonathan Zhu for his advice and technical suggestions, some of which have materialised in this thesis. I now understand what you meant when you told me that 'everything happens on one page'.

I am grateful to everyone who made my year-long excursion to Canberra a worthwhile and memorable one. Thanks to all the Honours students for their good humour, and thanks to those in our corner of the office – Ed, James, Junbang, Shawn and Yuyang – for the spontaneous conversations (and putting up with my talkativeness).

Lastly, a big thank you to my family for their unwavering love and support over the years.

Notation and Symbols

Unless otherwise specified, all manifolds are smooth and orientable without boundary, and all sections of vector bundles are smooth. We also adopt Einstein summation convention in which repeated indices are implicitly summed over. Expressions of the form $C = C(\alpha, \beta, \ldots)$ mean that C is a constant depending on α, β, etc. This constant can change from line to line but retains the most recently stated dependencies.

$\equiv$	is identically equal to
$\otimes$	tensor product
∇	covariant derivative or gradient operator
$\overline{\nabla}$	Euclidean covariant derivative or gradient operator
∇_i, $\overline{\nabla}_i$	covariant derivatives in the i-th coordinate direction with respect to a chart (resp. canonical coordinates in Euclidean space)
∇^k, $\overline{\nabla}^k$	k-th iterated covariant derivatives
Δ	Laplacian
$[f]$	weighted integrals over a hypersurface Σ, i.e. $\int_\Sigma f e^{-\frac{\lvert x\rvert^2}{4}}$ (used in §4)
$\lvert T\rvert$	norm of the tensor T on a hypersurface
$\lVert v\rVert_V$	norm of v in a normed vector space V
$*$	tensor contraction, possibly involving the metric
$\llcorner$	restriction of measure
$\langle\cdot,\cdot\rangle$	Riemannian metric or inner product
$(\star_{R,r,\ell,n})$	special quantity defined by (6.50) (used in §6)
A	second fundamental form
a.e.	almost everywhere
B_R, $B_R(x_0)$	Euclidean ball of radius R centred at the origin (resp. centred at x_0)
$C(\alpha,\beta,\ldots)$	a constant depending on $\alpha, \beta, \ldots$
$C^k(E)$, $C^{k,\alpha}(E)$	C^k sections, $C^{k,\alpha}$ Hölder sections of E
$\mathcal{C}_k$	set of all rotations of $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$ about the origin in $\mathbb{R}^{n+1}$ (used in §6)
d	de Rham differential or Fréchet derivative
div	divergence
$\mathcal{F}(\Sigma)$, $\mathcal{F}_\Sigma(u)$	$\mathcal{F}$-functional on a hypersurface Σ, or on the normal graph of u above Σ

ϕ	$-H + \frac{\langle x, \mathbf{n} \rangle}{2}$ (used in §6)
φ, $\widetilde{\varphi}$	MCF (resp. RMCF) as a one-parameter family of immersions
φ_t, $\widetilde{\varphi}_s$	timeslices of φ, $\widetilde{\varphi}$
g	Riemannian metric
$\Gamma(E)$, $\Gamma_c(E)$	smooth sections (resp. compactly supported smooth sections) of E
Γ_{ij}^k	Christoffel symbols
H	mean curvature
$\mathcal{H}^n$	n-dimensional Hausdorff measure
h_{ij}	components of the second fundamental form
L	typically the linearisations of $\mathcal{M}$, $\mathcal{M}_\Sigma$ etc. at 0
$L^p(E)$	L^p sections of E (Gaussian-weighted in §4 and §6)
$L^2(\mu)$, $L^2(\nu)$	describes an L^2 structure with inner product defined by paired integration against the measure μ (resp. ν)
$\mathcal{L}$	Ornstein-Uhlenbeck operator
M	typically a smooth, orientable n-dimensional manifold without boundary
MCF	mean curvature flow
$\mathcal{M}$, $\mathcal{M}_\Sigma$, etc.	Euler-Lagrange functional of $\mathcal{F}$, $\mathcal{F}_\Sigma$, possibly Gaussian-weighted
μ	Riemannian measure
NM	normal bundle of M
$\mathbb{N}$, $\mathbb{N}_0$	natural numbers excluding (resp. including) zero
$\mathbf{n}$	outward-pointing unit normal
ν	Gaussian measure on a hypersurface
$\mathcal{O}$	big O notation for asymptotic behaviour
RMCF	rescaled mean curvature flow
$R(\Sigma_T)$	entropy scale of the RMCF Σ_s at time $s = T$ (used in §6)
$\mathbf{r}_{\varepsilon,\ell,K}(\Sigma)$	cylindrical scale of the hypersurface Σ (used in §6)
S_r^n, S^n	n-sphere of radius r (resp. radius 1) centred at the origin in $\mathbb{R}^{n+1}$
Σ_s	RMCF as a family of compact, embedded hypersurfaces (used in §5 and §6)
TM, T^*M	tangent (resp. cotangent) bundle of M
τ	$\frac{A}{H}$ (used in §6)
$W^{k,p}(E)$	$W^{k,p}$ Sobolev sections of E
w_Γ	distance to the axis of the cylinder Γ (used in §6)

Contents

Chapter 1

Introduction

Under mean curvature flow, a smooth surface moves to decrease its area as rapidly as possible. This mechanism was first studied by material scientists almost a century ago to model grain growth in metal annealing [Sut28, vN52, Mul56], and has since found applications in image processing and cellular automata, among other things. Mathematical interest in MCF was invigorated in the late 1970s and 1980s, starting with Brakke's comprehensive monograph [Bra78] in the measure-theoretic setting, followed by seminal works of Huisken and others on classical solutions (e.g. [Hui84, Hui90, EH91]).

At its core, mean curvature flow is a nonlinear partial differential equation governing the motion of a hypersurface. Nonlinearity gives rise to singularities: solutions of the flow cannot, in general, exist smoothly for infinite time. Geometrically, this corresponds to the hypersurface collapsing due to curvature blowup, but even this phenomenon can take place in various ways; see Figure 1.1. Ergo, a large part of mean curvature flow revolves around one question: *what do singularities of the flow look like?*

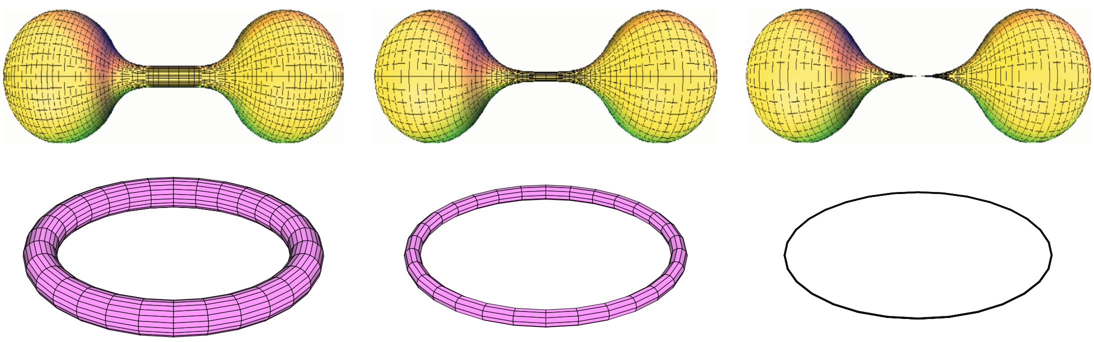

Figure 1.1: Different singularity modes for mean curvature flow in $\mathbb{R}^3$. Top (adapted from [CMP15]): a dumbbell develops a neckpinch, becoming singular at a point while the area remains positive. Bottom: a thin torus of revolution has becomes thinner, and the singularity locus is a circle. At the singular time, all surface area is lost.

To examine a singularity, one magnifies (or *blows up*) around the point where singular behaviour occurs to get a clearer picture of what is happening. One then tries to extract a limit hypersurface which would act as a geometric model of the singularity. Such limits are called *tangent flows*. By standard results [Hui90, Whi94, Ilm95], if the blowup is done at a suitable rate, then tangent flows always exist, and they belong to a special class of hypersurfaces called *shrinkers*. This is a powerful statement – not only does it provide a robust blowup method, but it also relates the study of singularities to the study of shrinkers. However, it has one major shortcoming: it only guarantees the existence of tangent flows in a subconvergent sense (think Arzelà–Ascoli). This means that multiple tangent flows could exist for a given singularity.

Suppose tangent flows were not unique, and we have multiple (possibly wildly different) tangent flows to model the same singularity. How could we then claim to really know what the singularity looks like? Which tangent flow models the singularity most accurately? Have we really magnified in way that gives an informative picture of how the singularity forms? Thus, in an ideal world, uniqueness of tangent flows is something that holds true. Whether or not this ideal is realised is a fundamental problem with far-reaching implications.

The earliest results on uniqueness of tangent flows are due to Gage–Hamilton–Grayson [GH86, Gra87] and Huisken [Hui84], respectively for mean curvature flow in $\mathbb{R}^2$ (the *curve shortening flow*) and for convex mean curvature flows in $\mathbb{R}^{n+1}$, $n \geq 2$. In fact, they proved that all tangent flows are spheres in these cases. More generally, tangent flows come in many different shapes and forms, making uniqueness much harder to prove. In a culmination of efforts made by [Sch14] and Colding–Minicozzi [CM15], alongside many precursors, uniqueness of tangent flows was finally proved for mean convex mean curvature flows in all dimensions. The ultimate goal of this thesis is to prove this result – a major accomplishment in the field.

Theorem 1.1 ([Sch14, CM15])**.** *Every tangent flow associated to a mean curvature flow of compact, embedded, mean convex hypersurfaces is unique.*

We will state a stronger, more precise version of this in Theorem 3.31, which additionally classifies all possible tangent flows that could arise.

In fact, the way Theorem 1.1/3.31 is proved gives an even more powerful result: all tangent flows in 'most' mean curvature flows are unique. This is made precise through the notion of *generic mean curvature flow* put forth by Colding and Minicozzi in [CM12]. We will not establish this result in full, but it will be discussed briefly in the final chapter.

1.1 Outline of the thesis

This thesis is written for a reader who is familiar with the fundamentals of Riemannian geometry and functional analysis, and has a passing acquaintance with partial differential equations. No prior knowledge about mean curvature flow is assumed. This thesis is also intended to be useful to specialists who seek to comprehend this small corner of the literature.

In §2, we lay out the necessary geometry and analysis background for later chapters. This also serves as an opportunity to declare our notation and conventions.

In §3, we introduce the mean curvature flow of Euclidean hypersurfaces and formalise the blowup procedure outlined above. We make the connection between tangent flows and shrinkers, and motivate the study of uniqueness of tangent flows.

In §4, we begin our quest to prove Theorem 1.1/3.31. After introducing some relevant machinery, we prove the classification result of [CM12]: the only embedded, mean convex shrinkers with polynomial volume growth are spheres, cylinders and hyperplanes.

In §5, we motivate the use of Łojasiewicz inequalities to study uniqueness of tangent flows. We then follow [Sim83a] and [Sim96] in proving the Łojasiewicz–Simon gradient inequality. Lastly, we use this to prove the uniqueness of all compact, embedded tangent flows [Sch14].

In §6, we prove the uniqueness of cylindrical tangent flows. This result is due to Colding and Minicozzi [CM15], but our treatment also draws from [Man14, CIM15, CM19b, Zhu20]. We acknowledge helpful suggestions provided to us by Jonathan Zhu.

In §7, we survey the current state of knowledge regarding uniqueness of tangent flows and other related aspects of mean curvature flow. We also outline some applications of the results presented and identify potential avenues for further work.

There are two appendices consisting of technical computations and estimates. These are well-known to experts, but are often employed in the literature without a reference. We included them to hopefully make the proofs accessible to a wider audience.

The key results presented in this thesis are not original. However, the road to establishing them is dotted with original contributions, chiefly in exposition, reorganisation of material, and added detail to proofs. Where there are gaps and/or inconsistencies in the original sources, we do our best to correct them. These corrections use ideas from other papers as well as ideas of our own. As a result, some of our statements and proofs are quite different from the original ones (this mostly occurs in §6). We will highlight these differences along the way.

Chapter 2

Preliminaries

Unless otherwise specified, all manifolds are smooth and orientable without boundary, and all sections of vector bundles are smooth. We also adopt Einstein summation convention in which repeated indices are implicitly summed over.

2.1 Geometry of Euclidean hypersurfaces

The concepts in this section are explained in depth in most Riemannian geometry textbooks, e.g. [Lee18]. See [ACGL20, §5] for a setup similar to ours but carried out in greater detail. Our sign conventions however follow Colding and Minicozzi in [CM12, CM15].

The main objects we study are immersed Euclidean submanifolds of unit codimension. Thus, we let M be an n-dimensional manifold immersed by a smooth map $\varphi : M \to \mathbb{R}^{n+1}$; that is, the differential $d\varphi|_p : T_pM \to T_{\varphi(p)}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ is injective for all $p \in M$. Oftentimes we will require φ to be an embedding, i.e. an immersion that is a homeomorphism onto its image. In both the immersed and embedded cases, we call $\varphi(M)$ a *hypersurface*.

Define a metric g on M by pulling back the Euclidean inner product $\langle \cdot, \cdot \rangle$ by φ, so that $g(X, Y) = \langle d\varphi(X), d\varphi(Y) \rangle$ for $X, Y \in \Gamma(TM)$. Often we also take $\langle \cdot, \cdot \rangle$ to also mean $g(\cdot, \cdot)$ when there is no ambiguity. Taking local coordinates $\{x^i\}_{i=1}^n$ around $p \in M$, we obtain a local tangent frame $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ and cotangent frame $\{dx^i\}_{i=1}^n$. Thus, g is given in components by

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle \frac{\partial \varphi}{\partial x^i}, \frac{\partial \varphi}{\partial x^j} \right\rangle,$$

where $\frac{\partial \varphi}{\partial x^i}$ stands for $d\varphi(\frac{\partial}{\partial x^i})$. More generally, a tensor field $T \in \Gamma(TM^{\otimes k} \otimes T^*M^{\otimes l})$ on M, also called a (k, l)-*tensor*, is expressed in components by

$$T = T^{i_1 \cdots i_k}_{j_1 \cdots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}.$$

We extend g to a metric on (k, l)-tensors by setting

$$g(S, T) = g_{i_1 p_1} \cdots g_{i_k p_k} g^{j_1 q_1} g^{j_l q_l} S^{i_1 \cdots i_k}_{j_1 \cdots j_l} T^{p_1 \cdots p_k}_{q_1 \cdots q_l},$$

where g^{ij} are the elements of the inverse of g when written as a matrix (g_{ij}). Every tensor thus has a norm $|T| = g(T,T)^{1/2}$. These quantities are independent of the choice of local chart.

The *Riemannian volume form* on M induced by g (and thus φ) is given locally by

$$\sqrt{\det(g_{ij})}\, dx^1 \wedge \ldots \wedge dx^n,$$

and the associated *Riemannian measure* on M is given locally by $\mu = \sqrt{\det(g_{ij})}\mathcal{L}^n$, where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$.

We will be integrating extrinsically defined quantities over M. For a smooth function $f : U \to \mathbb{R}$ with $\varphi(M) \subset U \subset \mathbb{R}^{n+1}$, define $\int_M f(x)\, d\mu$ by

$$\int_M f(x)\, d\mu = \int_M (f \circ \varphi)(p)\, d\mu = \int_{\varphi(M)} f\, d\mathcal{H}^n,$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$ (the second equality is the area formula; see e.g. [Sim83b, §8]). Thus, x on the left-hand side is understood to mean $\varphi(p)$. When considering a family of immersions $\{\varphi_t\}$ of M, we write $d\mu_t$ in place of $d\mu$ to clarify which measure is integrated against. We omit $d\mu$ when only one immersion is at play. In any case, the meanings of these notations will be clear from context.

The image of $d\varphi : TM \to \varphi^*T\mathbb{R}^{n+1}$ is the tangent subbundle of the pullback bundle $\varphi^*T\mathbb{R}^{n+1}$, whose fibrewise orthogonal complement is the *normal bundle* $NM = (d\varphi(TM))^\perp$. Hence,

$$\varphi^*T\mathbb{R}^{n+1} = d\varphi(TM) \oplus NM. \tag{2.1}$$

Since M is orientable, NM is a trivial bundle and we let $\mathbf{n} \in \Gamma(NM)$ be the outward-pointing unit normal vector field on M.

Denoting by $\overline{\nabla}$ the Euclidean connection (directional derivative) on $T\mathbb{R}^{n+1}$, the map φ induces a pullback connection ${}^\varphi\overline{\nabla}$ on $\varphi^*T\mathbb{R}^{n+1}$ defined by

$${}^\varphi\overline{\nabla} : TM \times \Gamma(\varphi^*T\mathbb{R}^{n+1}) \to \varphi^*T\mathbb{R}^{n+1}, \qquad {}^\varphi\overline{\nabla}_v(\varphi^*Z) = \overline{\nabla}_{d\varphi(v)}Z,$$

for any $Z \in \Gamma(T\mathbb{R}^{n+1})$. Now let $X, Y \in \Gamma(TM)$. Then $d\varphi(Y) \in \Gamma(\varphi^*T\mathbb{R}^{n+1})$ at least locally, so ${}^\varphi\overline{\nabla}_X(d\varphi(Y)) \in \Gamma(\varphi^*T\mathbb{R}^{n+1})$. By (2.1) and the nonvanishing of $\mathbf{n}$, we can uniquely decompose

$${}^\varphi\overline{\nabla}_X(d\varphi(Y)) = d\varphi(\nabla_X Y) + A(X,Y)\mathbf{n},$$

with $\nabla_X Y \in \Gamma(TM)$ and $A(X,Y) \in C^\infty(M)$, at least locally. Then ∇ defines an affine connection on TM, in fact the Levi-Civita connection of (M,g). Meanwhile, $A \in \Gamma(T^*M \otimes T^*M)$ defines the *second fundamental form* of M, which is a symmetric $(0,2)$-tensor. By convention, the components of A are denoted h_{ij}. Using the musical isomorphism $\sharp_p : T_p^*M \overset{\cong}{\to} T_pM$, we lift A to a $(1,1)$-tensor $A^\sharp$ with components $h^i_j = g^{ik}h_{kj}$. Being symmetric, $A^\sharp$ is orthogonally diagonalisable at each point with *minus* its eigenvalues called the *principal curvatures*. The *mean curvature* function $H \in C^\infty(M)$ is minus the trace of $A^\sharp$, that is

$$H = -h^i_i = -g^{ij}h_{ji} = -g^{ij}h_{ij}.$$

In other words, $H(p)$ is the sum of the principal curvatures at p.

Let Γ_{ij}^k denote the Christoffel symbols of ∇ in local coordinates. The *Weingarten relations* read

$$\frac{\partial^2\varphi}{\partial x^i \partial x^j} = \Gamma_{ij}^k \frac{\partial \varphi}{\partial x^k} + h_{ij}\mathbf{n}, \qquad \frac{\partial \mathbf{n}}{\partial x^j} = -h_{jl} g^{ls} \frac{\partial \varphi}{\partial x^s}. \tag{2.2}$$

Taking inner products with $\mathbf{n}$ in the left-hand equation gives a formula for h_{ij},

$$h_{ij} = \left\langle \mathbf{n}, \frac{\partial^2\varphi}{\partial x^i \partial x^j} \right\rangle = -\left\langle \frac{\partial \mathbf{n}}{\partial x^i}, \frac{\partial \varphi}{\partial x^j} \right\rangle. \tag{2.3}$$

The connection ∇ on TM naturally induces connections (also written ∇) on tensor bundles of TM and T^*M. We view ∇ as a *covariant derivative* operator taking (k,l)-tensors to $(k,l+1)$-tensors. For a tensor $T = T_{j_1\dots j_l}^{i_1\dots i_k}$, the components of ∇T are written $\nabla_s T_{j_1\dots j_l}^{i_1\dots i_k} = (\nabla T)_{sj_1\dots j_l}^{i_1\dots i_k}$. We can also form the *iterated covariant derivatives* ∇^m which take (k,l)-tensors to $(k,l+m)$-tensors. Likewise, the components of $\nabla^m T$ are written $\nabla_{s_1}\cdots\nabla_{s_m} T_{j_1\dots j_l}^{i_1\dots i_k} = (\nabla^m T)_{s_1\dots s_m j_1\dots j_l}^{i_1\dots i_k}$.

The *gradient* of a function $f : M \to \mathbb{R}$, *divergence* of a vector field $X \in \Gamma(TM)$ and *Laplacian* of a tensor T are defined respectively by

$$\begin{aligned} \nabla f = (df)^\sharp, \quad \text{i.e.} \quad g(\nabla f(p), v) = df_p(v) \quad \forall v \in T_pM,\ \forall p \in M,\\ \operatorname{div} X = \operatorname{tr} \nabla X = \nabla_i X^i,\\ \Delta T = \operatorname{tr} \nabla^2 T = g^{ij}\nabla_i \nabla_j T. \end{aligned}$$

The *Hessian* of f is the $(0,2)$-tensor $\nabla^2 f$.

Of great importance will be the *Codazzi equations* and *Simons' equation* [Sim68],

$$\nabla_i h_{jk} = \nabla_j h_{ki} = \nabla_k h_{ij}, \tag{2.4}$$

$$\Delta h_{ij} = -\nabla_i \nabla_j H - H h_{il} g^{ls} h_{sj} - |A|^2 h_{ij}. \tag{2.5}$$

We also need the *first variation formula* for area:

Proposition 2.1 (First variation). *Let $\varphi : M \to \mathbb{R}^{n+1}$ be an immersion, and $\varphi_t : M \to \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$ be a smooth one-parameter family of immersions with $\varphi_0 = \varphi$. Let $X = \frac{\partial \varphi_t}{\partial t}\big|_{t=0}$ be the variation vector field along M. If μ_t is the Riemannian measure on M induced by φ_t, then*

$$\left.\frac{d}{dt}\right|_{t=0} d\mu_t = H \langle X, \mathbf{n}\rangle\, d\mu. \tag{2.6}$$

If $K \subset M$ is compact, then since $\operatorname{Area}(K,t) = \mathcal{H}^n(\varphi_t(K)) = \int_K d\mu_t$*, this gives*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Area}(K,t) = \int_K H \langle X, \mathbf{n}\rangle\, d\mu. \tag{2.7}$$

The formula (2.7) shows that $-H\mathbf{n}$ is the negative $L^2(\mu)$-gradient of the Area functional. For all variations such that the variation field X has $L^2(\mu \llcorner K)$-norm $(\int_K H^2 d\mu)^{1/2}$, it is $X = -H\mathbf{n}$ that yields the fastest instantaneous decrease in the area of K. This is the starting point for the mean curvature flow, which is formally introduced in §3.

2.2 Analysis on Riemannian manifolds

In this section, (M,g) is a Riemannian manifold with Levi-Civita connection ∇. General references for the material of this section are [Nic07] and [Eva10].

2.2.1 L^p, Sobolev and Hölder spaces

Let E be a real vector bundle over M with metric h and compatible connection ∇. Note that ∇ stands for both the connection on TM and on E, but there will be no ambiguity in which connection ought to be used. The metrics g and h induce metrics on tensor bundles of the form $T^*M^{\otimes k} \otimes E$; we also denote these by h. Likewise, these tensor bundles have natural connections ∇ induced from the connections on TM and on E.

For $1 \leq p \leq \infty$ and a measurable section $u : M \to E$, define

$$\|u\|_{L^p(E)} = \begin{cases} \left\{\int_M |u|_h^p\right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in M} |u(x)|_h & \text{if } p = \infty, \end{cases}$$

where $|\cdot|_h = h(\cdot,\cdot)^{1/2}$, and integration takes place against the Riemannian measure of (M, g). We define $L^p(E)$ to be the Banach space of measurable sections of E for which this norm is finite, modulo sections which agree a.e.. The space $L^2(E)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{L^2(E)} = \int_M h(u, v).$$

Next, we define the Sobolev spaces. For $k \in \mathbb{N}$, let $(\nabla^k)^*$ be the L^2-*formal adjoint* of the operator ∇^k, which is defined by requiring that

$$\int_M h(\nabla^k u, \psi) = \int_M h(u, (\nabla^k)^* \psi) \tag{2.8}$$

for all compactly supported smooth sections $u \in \Gamma_c(E)$ and $\psi \in \Gamma_c(T^*M^{\otimes k} \otimes E)$. For example, the divergence operator is minus the L^2-formal adjoint of the gradient operator, since

$$\int_M g(\nabla f, X) = -\int_M f \operatorname{div} X, \qquad \forall f \in C_c^\infty(M) = \Gamma_c(M \times \mathbb{R}),\ X \in \Gamma_c(TM), \tag{2.9}$$

by Stokes' Theorem (recall M has no boundary).

We use this to define weak derivatives. Let $L^1_{loc}(E)$ be the space of locally integrable L^1 sections of E, i.e. those in $L^1(E|_K)$ for every compact $K \subset M$. In close analogy to (2.8), for $u \in L^1_{loc}(E)$ and $v \in L^1_{loc}(T^*M^{\otimes k} \otimes E)$, we say that v is the k-th *weak derivative* of u if

$$\int_M h(v, \psi) = \int_M h(u, (\nabla^k)^* \psi), \qquad \forall \psi \in \Gamma_c(T^*M^{\otimes k} \otimes E).$$

It can be shown that a weak derivative is unique if it exists, so we write $\nabla^k u$ for the k-th weak derivative of u. For $k \in \mathbb{N}$ and $p \in [1, \infty)$, the Sobolev space $W^{k,p}(E)$ consists of sections $u \in L^p(E)$ such that $\nabla^j u$ exists and belongs to $L^p(T^*M^{\otimes j} \otimes E)$ for each $j = 1, \ldots, k$. It is a Banach space with the norm

$$\|u\|_{W^{k,p}(E)} = \left\{\sum_{j=0}^k \left\|\nabla^j u\right\|^p_{L^p(T^*M^{\otimes j} \otimes E)}\right\}^{1/p},$$

where the induced metric h is used to evaluate the $L^p(T^*M^{\otimes j} \otimes E)$ norms. We also need that $W^{k,2}(E)$ is a Hilbert space for each k, with inner product

$$\langle u, v\rangle_{W^{k,2}(E)} = \sum_{j=0}^{k} \left\langle \nabla^j u, \nabla^j v\right\rangle_{L^2(T^*M^{\otimes j}\otimes E)}.$$

Finally, we define C^k and Hölder spaces. For $k \in \mathbb{N}_0$, let $C^k(E)$ be the space of sections of E whose (strong) derivatives of order $\leq k$ exist, and are all bounded and continuous. It is a Banach space with norm

$$\|u\|_{C^k(E)} = \sum_{j=0}^{k} \left\|\nabla^j u\right\|_{L^\infty(T^*M^{\otimes j}\otimes E)}.$$

Assuming M is complete and has positive injectivity radius ρ, let $\rho_0 = \min\{1, \rho\}$ and define (for $0 < \alpha \leq 1$) the α-Hölder seminorm of a section u by

$$[u]_{\alpha;E} = \sup_{\substack{x,y\in M\\ 0<d(x,y)<\rho_0}} \frac{|u(x) - P_{x,y}u(y)|_h}{d(x,y)^\alpha},$$

where $P_{x,y}$ is the (E,∇)-parallel transport from the fibre above y to the fibre above x, and $d(\cdot,\cdot)$ is the distance on M. Then $C^{k,\alpha}(E)$ is the Banach space of sections in $C^k(E)$ for which the norm

$$\|u\|_{C^{k,\alpha}(E)} = \|u\|_{C^k(E)} + [\nabla^k u]_{\alpha;T^*M^{\otimes k}\otimes E}$$

is finite.

When we write $L^p(M)$, $W^{k,p}(M)$ or $C^{k,\alpha}(M)$, we are referring to spaces of real-valued functions on M, i.e. the spaces defined above where $E = M \times \mathbb{R}$ is the trivial bundle. We will simply write L^p, $W^{k,p}$ and $C^{k,\alpha}$ when there is no ambiguity as to which vector bundle is at use.

2.2.2 Calculus on Banach spaces

Let U and V be Banach spaces, and denote by $B(U,V)$ the Banach space of bounded linear operators from U to V endowed with the operator norm. The *linearisation* (or *Fréchet derivative*) of a map $f : U \to V$ at $u \in U$, if it exists, is a bounded linear operator $df|_u \in B(U,V)$ such that

$$f(u+h) = f(u) + df|_u(h) + o(\|h\|_U).$$

The *Hessian* of f at u is the linearisation of the map $U \to B(U,V)$, $x \mapsto df|_x$ at u. Thus it is an operator $d^2f|_u \in B(U, B(U,V))$, and it is customary to make the identification $B(U,B(U,V)) \cong B(U\times U, V)$ whereby $d^2f|_u$ becomes a bilinear map. Defining higher derivatives similarly, one has Taylor's theorem which says that if f is $n+1$ times continuously differentiable in a neighbourhood of u, then for $\|h\|_U$ small there exists $\varepsilon \in (0,1)$ such that

$$f(u+h) = f(u) + \sum_{k=1}^{n} \frac{1}{k!} d^k f|_u(h^k) + \frac{1}{(n+1)!} d^{n+1} f|_{u+\varepsilon h}(h^{n+1}), \tag{2.10}$$

where h^k stands for the k-tuple $(h, h, \dots, h)$.

Let ∇ be the gradient operator on (M,g). We encounter functionals $\mathcal{E} : C^1(M) \to \mathbb{R}$ of the form

$$\mathcal{E}(u) = \int_M E(p, u(p), \nabla u(p)),$$

where E is a smooth function of (p,q,z) where $p \in M$, $q \in \mathbb{R}$, and $z \in T_pM$.[1] The *Euler–Lagrange functional* of $\mathcal{E}$ with respect to an L^2 structure on M is the operator $\mathcal{M} : C^2(M) \to C^0(M)$ defined by

$$-\langle \mathcal{M}u, v\rangle_{L^2} = \frac{d}{ds}\Big|_{s=0} \mathcal{E}(u+sv), \quad \forall v \in C^2(M), \tag{2.11}$$

that is the negative L^2-gradient functional of $\mathcal{E}$. For any $u \in C^2(M)$, the linearisation $L_u = D\mathcal{M}|_u$ is a symmetric operator (see §2.2.3) since

$$\langle L_u v, w\rangle_{L^2} = \frac{d}{ds}\Big|_{s=0} \langle \mathcal{M}(u+sv), w\rangle_{L^2} = \frac{d^2}{dsdt}\Big|_{s,t=0} \mathcal{E}(u+sv+tw) = \langle v, L_u w\rangle_{L^2}. \tag{2.12}$$

Appendix A contains some calculations regarding $\mathcal{M}$ and L which find use in the thesis.

2.2.3 Ordinary and partial differential equations

Let L be a second-order linear partial differential operator on M. That is, L acts on a (weakly) twice-differentiable function u locally by

$$Lu = a^{ij}\nabla_i\nabla_j u + b^\alpha \nabla_\alpha u + cu,$$

where a^{ij}, b^α and c are coefficient functions in a chart, and $a^{ij} = a^{ji}$. We say that L is *(uniformly) elliptic* if there exists $C > 0$ such that for all $p \in M$ and $\xi \in T_p^*M$,

$$a^{ij}(p)\xi_i\xi_j \geq C|\xi|^2,$$

where $|\xi|$ uses the norm on T^*M induced by the metric on M. It can be shown that the left-hand side is chart-independent, so ellipticity is a well-defined notion.

As in (2.8), L has a unique *formal adjoint* L^*, defined by requiring that

$$\int_M (Lu)v = \int_M u(L^*v), \qquad \forall v \in C_c^\infty(M).$$

If $L = L^*$, then L is *symmetric*. We now state some standard results in elliptic theory. For Theorems 2.2-2.4 below, we assume that M is compact without boundary and L is symmetric and uniformly elliptic with smooth coefficients.

Theorem 2.2. *L has discrete spectrum with finite multiplicity on $W^{2,2}$, and $W^{2,2}$ is spanned by a complete basis of smooth L^2-orthonormal eigenfunctions of L.*

The next two theorems concern solutions u to the elliptic equation $Lu = f$, where f is a given function. This includes *weak solutions*, which are those which satisfy

$$\int_M fv = \int_M u(Lv), \qquad \forall v \in C^\infty(M).$$

We write $\mathcal{K}$ for the kernel of L, and $\mathcal{K}^\perp$ for the L^2-orthogonal complement of $\mathcal{K}$.

[1] That is, $(q,z) \mapsto E(p,q,z)$ is smooth for each $p \in M$, and $(p,q) \to E(p,q,X_p)$ is smooth whenever $X \in \Gamma(TM)$.

Theorem 2.3. *Let $f \in L^2$. Then $Lu = f$ has a weak solution $u \in W^{2,2}$ if and only if $f \in \mathcal{K}^\perp$. If a solution exists, then there is a unique weak solution in $W^{2,2} \cap \mathcal{K}^\perp$.*

Theorem 2.4 (Schauder estimates)**.** *Suppose u and f are such that $Lu = f$, and $\alpha \in (0,1)$.*

(i) If $f \in C^{k,\alpha}$, then $u \in C^{k+2,\alpha}$ and there exists a constant C such that

$$\|u\|_{C^{k+2,\alpha}} \leq C(\|f\|_{C^{k,\alpha}} + \|u\|_{C^{0,\alpha}}).$$

(ii) If $u \in C^{k+2,\alpha} \cap \mathcal{K}^\perp$, then there exists a constant C such that

$$\|u\|_{C^{k+2,\alpha}} \leq C \|f\|_{C^{k,\alpha}}.$$

The same conclusions hold if we replace $C^{k+2,\alpha}$ and $C^{k,\alpha}$ with $W^{k+2,2}$ and $W^{k,2}$, respectively.

For the next theorem, we make two changes. Firstly, we allow u, the coefficient functions of L, and the metric g to be time-dependent. This causes the spatial derivatives ∇_i to also be time-dependent, as they depend on g. Secondly, we allow L to have a nonlinear reaction term. Our statement comes from [Man11, Theorem 2.1.1] and [ACGL20, Corollary 1.4].

Theorem 2.5 (Parabolic maximum principles)**.** *Let $g(t)$, $t \in [0,T)$ be a family of Riemannian metrics on M, smoothly time-varying in the sense that the components $g_{ij}(p,t)$ are smooth functions in some (hence any) local coordinates. Suppose $u : M \times [0,T) \to \mathbb{R}$ is smooth and*

$$\partial_t u \leq a^{ij}\nabla_i\nabla_j u + b^\alpha \nabla_\alpha u + F(u),$$

where a^{ij}, b^α are smooth functions of $(p,t) \in M \times [0,T)$ and $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Moreover, suppose that $a^{ij} = a^{ji}$, and there exists $C > 0$ such that $a^{ij}(p,t)\xi_i\xi_j \geq C|\xi|^2$ for all $p \in M$, $\xi \in T_p^*M$ and $t \in [0,T)$. Then the following hold:

(i) The function $u_{\max}(t) = \max_{p\in M} u(p,t)$ is locally Lipschitz, hence differentiable at almost every time $t \in [0,T)$ and at every differentiability time,

$$\frac{du_{\max}(t)}{dt} \leq F(u_{\max}(t)).$$

(ii) If $T' \leq T$ and $h : [0,T') \to \mathbb{R}$ solves the initial value problem

$$h'(t) = F(h(t)), \quad h(0) = u_{\max}(0),$$

then $u \leq h$ in $M \times [0,T')$.

(iii) If M is connected and $u_{\max}(\tau) = h(\tau)$ for some $\tau \in (0,T')$, then $u = h$ in $M \times [0,\tau]$.

The same conclusions hold if we replace $u_{\max}$ with $u_{\min}$ and reverse the appropriate inequalities.

Remark 2.6. Part (iii) of the above theorem is the *strong maximum principle*. For this it suffices that M is connected and the maximum is attained; compactness is not required.

Finally, we need a basic result concerning ordinary differential inequalities.

Lemma 2.7 (Grönwall's lemma)**.** *Let u and β be real-valued continuous functions defined on an interval $[a,b)$ where $a < b \leq \infty$. If u is differentiable a.e. on (a,b) with $u'(t) \leq \beta(t)u(t)$, then*

$$u(t) \leq u(a) \exp\left\{\int_a^t \beta(s)\, ds\right\}, \qquad \forall t \in (a,b).$$

Chapter 3

Mean Curvature Flow and Blowups

In this chapter, we introduce the mean curvature flow and the blowup procedure for singularity analysis. We have reorganised what is available in various books and lecture notes (e.g. [Man11, ACGL20]), presenting only what is needed for our forthcoming analysis. We will not give full proofs for all statements in this chapter; these are found in the aforementioned texts. We conclude by discussing the uniqueness of tangent flows problem and its importance in singularity analysis. We also recommend [CMP15] for a wonderful non-technical survey of mean curvature flow.

3.1 The mean curvature flow

Mean curvature flow (MCF) is a dynamical system whereby a hypersurface evolves to locally decrease its area as rapidly as possible. If M is an n-dimensional manifold and φ is a time-dependent immersion of M into $\mathbb{R}^{n+1}$, then by Proposition 2.1 the area of M locally decreases as rapidly as possible when the variation field is $-H\mathbf{n}$. We use this to define the mean curvature flow of M.

Definition 3.1. Let $I \subset \mathbb{R}$ be an interval. A one-parameter family of immersions $\varphi : M \times I \to \mathbb{R}^{n+1}$ is a solution to *mean curvature flow* (MCF) if it satisfies

$$\frac{\partial \varphi}{\partial t}(p,t) = -H(p,t)\mathbf{n}(p,t). \tag{3.1}$$

The true objects of interest are the images $\varphi_t(M) = \varphi(M,t)$, which are invariant under tangential diffeomorphisms of M. For this reason, we will allow $\frac{\partial \varphi}{\partial t}$ to possess a tangential component in addition to (3.1), and this leads to a more general definition of MCF.

Definition 3.2. If $\varphi : M \times I \to \mathbb{R}^{n+1}$ is a family of immersions satisfying

$$\left\langle \frac{\partial \varphi}{\partial t}(p,t), \mathbf{n}(p,t) \right\rangle = -H(p,t), \tag{3.2}$$

then we still consider φ to be a solution to MCF. Taking this further, a one-parameter family of hypersurfaces $\{\Sigma_t\}_{t\in I}$ in $\mathbb{R}^{n+1}$ is said to *flow by MCF* if there are immersions $\varphi_t : M \to \mathbb{R}^{n+1}$ of a smooth n-manifold M satisfying $\varphi_t(M) = \Sigma_t$ and (3.2).

Minimal hypersurfaces (those with $H \equiv 0$) are stationary under MCF. As for nontrivial solutions, we can explicitly write down a few:

Example 3.3. Let S^n_r be the n-sphere of radius r. Then the shrinking spheres $\{S^n_{\sqrt{r^2-2nt}}\}_{t\in[0,r^2/2n)}$ flow by MCF. To see this, one can use the explicit parametrisation

$$\varphi : S^n_1 \times \left[0, \frac{r^2}{2n}\right) \to \mathbb{R}^{n+1}, \qquad \varphi(p,t) = \sqrt{r^2 - 2nt} \cdot p.$$

Equation (3.1) is easily verified since $H(p,t) = n/\sqrt{r^2 - 2nt}$ and $\mathbf{n}(p,t) = p$. The maximal time of existence is $\frac{r^2}{2n}$, when the sphere collapses to a point (one says the flow becomes *extinct*). Generalising this, the shrinking cylinders $\{S^k_{\sqrt{r^2-2kt}} \times \mathbb{R}^{n-k}\}_{t\in[0,r^2/2k)}$ flow by MCF, with

$$\varphi : S^k_1 \times \mathbb{R}^{n-k} \times \left[0, \frac{r^2}{2k}\right) \to \mathbb{R}^{k+1} \times \mathbb{R}^{n-k} \cong \mathbb{R}^{n+1}, \qquad \varphi(p,x,t) = (\sqrt{r^2 - 2kt} \cdot p, x).$$

Example 3.4. The *Grim Reaper* solution is the one-parameter family of translating plane curves

$$\left\{(x,y) \in \mathbb{R}^2 \mid y = t + \log \sec x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}_{t\in\mathbb{R}}.$$

Being defined for all $t \in \mathbb{R}$, this is an *eternal* solution.

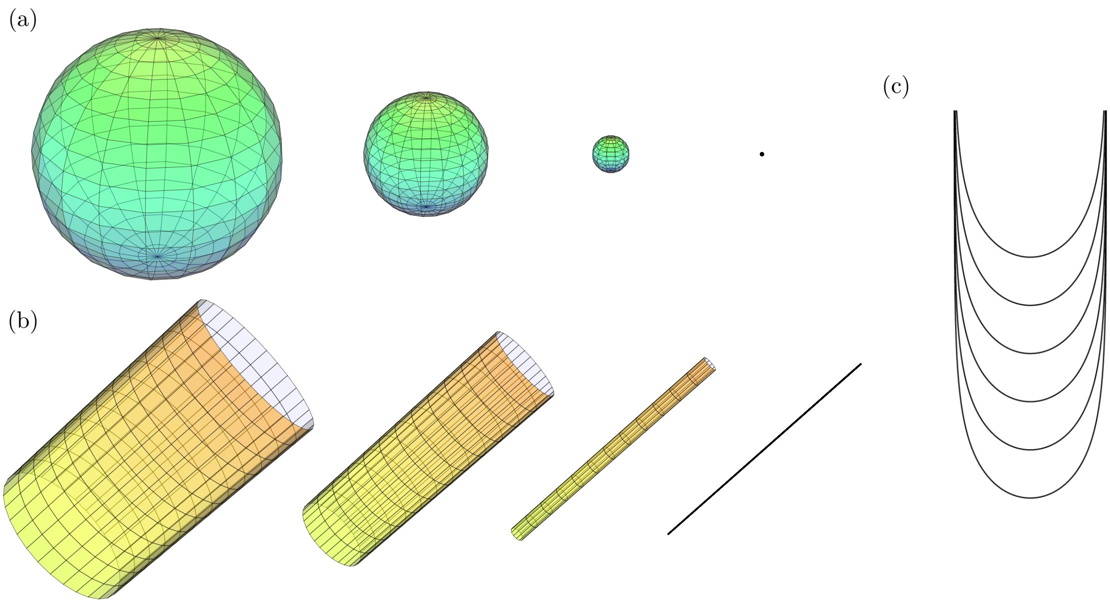

Figure 3.1: Simple solutions to mean curvature flow. (a) A shrinking sphere, (b) a shrinking cylinder, (c) the translating Grim Reaper.

Besides these, explicit solutions are in shortage. This urges the development of a short-time existence theory, which would guarantee the existence of a solution given an initial immersion of M. Note that the MCF equation (3.1) is formally similar to the heat equation, since by (2.2),

$$\frac{\partial \varphi}{\partial t} = -H\mathbf{n} = g^{ij} h_{ij} \mathbf{n} = g^{ij} \left(\frac{\partial^2 \varphi}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \varphi}{\partial x^k} \right) = g^{ij} \nabla_i \nabla_j \varphi = \Delta \varphi. \tag{3.3}$$

However, Δ is time-dependent since its coefficients g^{ij} depend on φ_t, and this renders the PDE (3.3) degenerate parabolic (see [Zhu02] or [ACGL20]). Standard parabolic theory cannot apply to give short-time existence, but it turns out the degeneracies all incarnate in the tangential directions. Using this and the *DeTurck trick*, we can turn the more general MCF equation (3.2) into a parabolic system, which gives short-time existence at least when M is compact.

Theorem 3.5 (Short time existence). *If M is compact and $\psi : M \to \mathbb{R}^{n+1}$ is an immersion, then there exists a solution φ to (3.2) defined on a positive interval with $\varphi(\cdot, 0) = \psi$. Moreover, φ is unique up to reparametrisation at each time. We call φ the MCF with initial condition ψ.*

The MCF 'of' a compact, embedded hypersurface Σ is the MCF with initial condition ι, where $\iota : \Sigma \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion. Next, we record two foundational theorems and a corollary.

Theorem 3.6. *The MCF of an embedded hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ remains embedded at all times.*

Theorem 3.7 (Avoidance principle). *If Σ_1 and Σ_2 are disjoint compact hypersurfaces in $\mathbb{R}^{n+1}$, then the MCFs of Σ_1 and Σ_2 remain disjoint whenever both flows are defined.*

Corollary 3.8. *The MCF of a compact hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ exists only up to a finite time.*

Proof. Let S^n_r be a large sphere centred at the origin strictly enclosing Σ. By Example 3.3, the MCF of S^n_r exists for a maximal time interval of length $r^2/2n$. By Theorem 3.7, the MCF of Σ ceases to exist at or before this amount of time has elapsed. □

This shows that singularities are inevitable in compact MCF (i.e. when M is compact), so we have no choice but to study them. The central object is the *singular set* $\mathcal{S}$, the set of points in $\mathbb{R}^{n+1}$ where singularities appear (we will define this precisely later). For shrinking spheres and cylinders in $\mathbb{R}^3$, $\mathcal{S}$ is a point and a line respectively. Does $\mathcal{S}$ always take on a nice geometry, or could it be structured wildly? What is its dimension and degree of regularity? We can also ask about local properties of $\mathcal{S}$: for example, how does an MCF behave near a singular point $\widehat{p} \in \mathcal{S}$ as the singular time is approached? What implications does this have on the geometry of $\mathcal{S}$ near $\widehat{p}$? These questions guide much of today's research in singularities of the flow.

Shrinking spheres and cylinders are prototypes of *self-shrinking MCFs* – solutions that contract around a point in $\mathbb{R}^{n+1}$ up to reparametrisation. Because (3.2) is invariant under rigid spacetime translations, understanding all self-shrinking MCFs is a matter of understanding the hypersurfaces which generate a shrinking flow around the origin for time one. Such hypersurfaces are called *shrinkers*, and they are expected to exist in large numbers; nontrivial constructions in $\mathbb{R}^3$ include Angenent's shrinking torus [Ang92] and those considered by Nguyen [Ngu14]. The eager search for shrinkers traces back to a classic result of Huisken ([Hui90]; see Theorem 3.27), which reduces the study of singularities in MCF to that of shrinkers. Hence, it is beneficial to have multiple characterisations of shrinkers, which the next theorem establishes.

Theorem 3.9. *The following are equivalent for an immersion $\varphi_{-1} : M \to \mathbb{R}^{n+1}$ with H not identically zero.*

(i) φ_{-1} immerses a shrinker. That is, there exists an MCF φ satisfying

$$\varphi : M \times [-1, 0) \to \mathbb{R}^{n+1}, \qquad \varphi(p, t) = f(t)\varphi_{-1}(p), \tag{3.4}$$

where f is a smooth positive function with $f(-1) = 1$, $f'(t) \leq 0$ and $\lim_{t \nearrow 0} f(t) = 0$.

(ii) There exists an MCF $\varphi : M \times [-1, 0) \to \mathbb{R}^{n+1}$ satisfying $\varphi(p,t) = \sqrt{-t} \cdot \varphi_{-1}(p)$.

(iii) The identity $H(p) = \frac{\langle \varphi_{-1}(p), \mathbf{n}(p) \rangle}{2}$ holds for all $p \in M$.

Proof. (i) $\Leftrightarrow$ (ii): Suppose (i) holds. By (3.2), for any $p \in M$ we have

$$\left\langle f'(t)\varphi_{-1}(p), \mathbf{n}(p,t) \right\rangle = \langle \partial_t \varphi(p,t), \mathbf{n}(p,t) \rangle = -H(p,t) = -\frac{H(p,-1)}{f(t)},$$

so $\left\langle \partial_t (f(t)^2), \mathbf{n}(p,t) \right\rangle = \langle f(t) f'(t) \varphi_{-1}(p), \mathbf{n}(p,t) \rangle = -H(p,-1)$. Note that $\mathbf{n}(p,t)$ is actually independent of t by (3.4). Assuming p is a point where $H(p,-1) \neq 0$, we must then have $\partial_t(f(t)^2)$ independent of t as well. Using the Taylor series of f^2 around $t = -1$, we get that f is of the form $f(t) = \sqrt{c + \alpha(t+1)}$ for some constants c, α. Substituting $f(-1) = 1$ and $\lim_{t \nearrow 0} f(t) = 0$ gives $f(t) = \sqrt{-t}$, which is (ii). Meanwhile, (ii) $\Rightarrow$ (i) is immediate.

(ii) $\Leftrightarrow$ (iii): If (ii) holds, then by (3.2) we have

$$H(p,t) = -\langle \partial_t \varphi(p,t), \mathbf{n}(p,t) \rangle = \left\langle \frac{1}{2\sqrt{-t}} \cdot \varphi_{-1}(p), \mathbf{n}(p,t) \right\rangle.$$

Putting $t = -1$ yields (iii). Conversely, if (iii) holds, then setting $\varphi(p,t) = \sqrt{-t} \cdot \varphi_{-1}(p)$ we have $\mathbf{n}(p,t) = \mathbf{n}(p,-1)$, so

$$\langle \partial_t \varphi(p,t), \mathbf{n}(p,t) \rangle = \left\langle -\frac{1}{2\sqrt{-t}} \cdot \varphi_{-1}(p), \mathbf{n}(p,-1) \right\rangle = -\frac{1}{\sqrt{-t}} H(p,-1) = -H(p,t),$$

It follows that φ is an MCF with initial condition $\varphi(p,-1) = \varphi_{-1}(p)$. $\square$

Remark 3.10. Although Theorem 3.9 assumes H does not identically vanish, the minimal hypersurfaces $H \equiv 0$ are easily handled in our applications. We will therefore call any hypersurface satisfying $H(p) = \frac{\langle \varphi_{-1}(p), \mathbf{n} \rangle}{2}$ a shrinker.

For an embedded hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, the implied immersion is the inclusion map, so by the above theorem, Σ is a shrinker if and only if $H = \frac{\langle x, \mathbf{n} \rangle}{2}$ for all $x \in \Sigma$.

3.2 Evolution equations and consequences

In this section, we derive evolution equations of key quantities and use these to deduce basic properties of MCF. To make sense of the next lemma, if T is a tensor with time-varying components $T^{i_1 \cdots i_k}_{j_1 \cdots j_l}(t)$, then its time derivative is a tensor $\partial_t T$ with components $(\partial_t T)^{i_1 \cdots i_k}_{j_1 \cdots j_l} = \partial_t (T^{i_1 \cdots i_k}_{j_1 \cdots j_l})$. The notation $S * T$ between tensors refers to a contraction of the tensors, possibly involving the metric. We later need that $|S * T| \leq C|S||T|$, where C depends only on n and the structure of the contraction, and in particular *not* on $p \in M$.

Lemma 3.11. *A mean curvature flow φ obeys the following evolution equations:*

$$\partial_t g = 2HA, \quad \partial_t g^{ij} = -2Hh^{ij}, \quad \partial_t \mathbf{n} = \nabla H, \tag{3.5}$$

$$\partial_t A = -\nabla^2 H + A^2 H = \Delta A + 2A^2 H + |A|^2 A, \tag{3.6}$$

$$\partial_t |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4, \tag{3.7}$$

$$\partial_t H = \Delta H + |A|^2 H, \tag{3.8}$$

$$\partial_t |\nabla^k A|^2 = \Delta |\nabla^k A|^2 - 2|\nabla^{k+1} A|^2 + \sum_{p+q+r=k|p,q,r\in\mathbb{N}} \nabla^p A * \nabla^q A * \nabla^r A * \nabla^k A. \tag{3.9}$$

Here we define $h^{ij} = g^{ik} g^{jl} h_{kl}$ and $(A^2)_{ij} = h_{il} g^{ls} h_{sj}$.

These standard computations can be found in [Zhu02] and [Man11] (beware of sign conventions). We derive (3.5), (3.6) and (3.8), in fact in a more general setting:

Lemma 3.12. *Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be an immersion, and $\{\varphi_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a normal variation of φ_0 where each $\varphi_t : M \to \mathbb{R}^{n+1}$ is also an immersion. Write $\frac{\partial \varphi_t}{\partial t}\big|_{t=0} = f\mathbf{n}$ for the variation field along M, where $f : M \to \mathbb{R}$ is smooth. Then the following evolution equations hold at $t = 0$:*

$$\partial_t g = -2fA, \quad \partial_t g^{ij} = 2fh^{ij}, \quad \partial_t \mathbf{n} = -\nabla f, \tag{3.10}$$

$$\partial_t A = \nabla^2 f - A^2 f, \tag{3.11}$$

$$\partial_t H = -\Delta f - |A|^2 f. \tag{3.12}$$

Proof. In this proof, all partial derivatives with respect to t are taken at $t = 0$. Keeping in mind the Weingarten relations (2.2), the metric evolves by

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \varphi_t}{\partial x^i}, \frac{\partial \varphi_t}{\partial x^j} \right\rangle = \left\langle \frac{\partial (f\mathbf{n})}{\partial x^i}, \frac{\partial \varphi_0}{\partial x^j} \right\rangle + \left\langle \frac{\partial \varphi_0}{\partial x^i}, \frac{\partial (f\mathbf{n})}{\partial x^j} \right\rangle \\
&= -\left\langle f h_{il} g^{ls} \frac{\partial \varphi_0}{\partial x^s}, \frac{\partial \varphi_0}{\partial x^j} \right\rangle - \left\langle \frac{\partial \varphi_0}{\partial x^i}, f h_{jl} g^{ls} \frac{\partial \varphi_0}{\partial x^s} \right\rangle = -f h_{il} g^{ls} g_{sj} - f h_{jl} g^{ls} g_{is} \\
&= -2f h_{ij},
\end{aligned}$$

that is $\partial_t g = -2fA$. Next, differentiate $g^{ij} = g^{is} g_{sl} g^{lj}$ to get

$$\partial_t g^{ij} = (\partial_t g^{is}) g_{sl} g^{lj} + g^{is} (\partial_t g_{sl}) g^{lj} + g^{is} g_{sl} (\partial_t g^{lj}) = 2\partial_t g^{ij} + g^{is} (\partial_t g_{sl}) g^{lj}.$$

Hence,

$$\partial_t g^{ij} = -g^{is} (\partial_t g_{sl}) g^{lj} = 2f g^{is} h_{sl} g^{lj} = 2f h^{ij}. \tag{3.13}$$

The $\frac{\partial \varphi_0}{\partial x^i}$-component of the time derivative of $\mathbf{n}$ is

$$\left\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial \varphi_0}{\partial x^i} \right\rangle = -\left\langle \mathbf{n}, \frac{\partial^2 \varphi_0}{\partial t \partial x^i} \right\rangle = -\left\langle \mathbf{n}, \frac{\partial (f\mathbf{n})}{\partial x^i} \right\rangle = -\frac{\partial f}{\partial x^i},$$

so $\partial_t \mathbf{n} = -\nabla f$. This proves the three equations in (3.10). Using these and (2.3), we compute

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= \frac{\partial}{\partial t}\left\langle \mathbf{n}, \frac{\partial^2 \varphi_t}{\partial x^i \partial x^j} \right\rangle = \left\langle -\nabla f, \frac{\partial^2 \varphi_0}{\partial x^i \partial x^j} \right\rangle + \left\langle \mathbf{n}, \frac{\partial^2 (f\mathbf{n})}{\partial x^i \partial x^j} \right\rangle \\
&= -\left\langle \frac{\partial f}{\partial x^l} \frac{\partial \varphi_0}{\partial x^s} g^{ls}, \Gamma^k_{ij} \frac{\partial \varphi_0}{\partial x^k} + h_{ij}\mathbf{n} \right\rangle + \frac{\partial^2 f}{\partial x^i \partial x^j} + \left\langle \mathbf{n}, f \frac{\partial}{\partial x^j}\left(-h_{il} g^{ls} \frac{\partial \varphi_0}{\partial x^s} \right) \right\rangle \qquad (3.14) \\
&= -\frac{\partial f}{\partial x^l} \Gamma^k_{ij} g^{ls} g_{sk} + \frac{\partial^2 f}{\partial x^i \partial x^j} - f h_{il} g^{ls} h_{sj} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} - f h_{il} g^{ls} h_{sj} \\
&= \nabla_i \nabla_j f - f h_{il} g^{ls} h_{sj},
\end{aligned}$$

which is the ij-component of $\nabla^2 f - fA^2$. This is (3.11). Finally, (3.13) and (3.14) give

$$\partial_t H = \partial_t(-g^{ij}h_{ij}) = -2fh^{ij}h_{ij} - g^{ij}(\nabla_i\nabla_j f - fh_{il}g^{ls}h_{sj}) = -2f|A|^2 - \Delta f + f|A|^2 = -\Delta f - |A|^2 f,$$

which is (3.12). □

Partial proof of Lemma 3.11. Equations (3.5), (3.6) and (3.8) are immediate from Lemma 3.12 by substituting $f = -H$ and shifting in time to get the evolution equations for all times t. The second equality in (3.6) follows from the first using Simons' equation (2.5). □

The next three results highlight some geometric consequences of Lemma 3.11.

Proposition 3.13. *If the initial hypersurface is compact and mean convex (i.e. $H \geq 0$ everywhere), then every timeslice of its MCF has $H \geq 0$. (Thus we may speak of 'mean convex MCF'.)*

Proof. For a contradiction, suppose $H_{\min} < 0$ at some time. By the continuity of $H_{\min}$ and the initial mean convexity, there is a time interval (t_1, t_2) in which $H_{\min}(t) < 0$ and $H_{\min}(t_1) = 0$. Let $|A|^2 \leq C$ in this interval; this is possible by the compactness of M. Combining with (3.8) gives $\partial_t H \leq \Delta H + CH$, so by the maximum principle (Theorem 2.5),

$$\partial_t H_{\min} \geq CH_{\min}, \qquad t \in (t_1, t_2) \text{ a.e.}$$

If $s \in (t_1, t_2)$, then applying Lemma 2.7 in the interval $[s, t_2)$ gives $H_{\min}(t) \geq e^{C(t_2 - s)} H_{\min}(s)$. Sending $s \searrow t_1$, we get $H_{\min}(t) \geq 0$ for all $t \in (t_1, t_2)$ which is a contradiction. □

The next result says that when a compact MCF approaches its maximal time T (which is finite by Corollary 3.8), it experiences curvature blowup. We skip the proof, instead referring readers to the references given at the beginning of this chapter.

Theorem 3.14 (Long time existence)**.** *If M is compact and $\varphi : M \times I \to \mathbb{R}^{n+1}$ is the MCF of M defined up to a maximal time $T < \infty$, then*

$$\lim_{t \nearrow T} \max_{p \in M} |A(p,t)| = \infty.$$

The next proposition quantifies Theorem 3.14, giving a lower bound for the curvature blowup rate. This bound is sharp; it is attained by shrinking spheres and cylinders.

Proposition 3.15. *Let the MCF of a compact hypersurface be defined up to a maximal time T. Then*

$$\max_{p \in M} |A(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}.$$

Proof. Write $|A|^2_{\max} = \max_{p \in M} |A(p)|^2$. The evolution equation (3.7) for $|A|^2$ implies

$$\partial_t |A|^2_{\max} \leq 2|A|^4_{\max} \tag{3.15}$$

at all times, since $\Delta|A|^2 \leq 0$ at the spatial maximum of $|A|^2$. Note that $|A|^2_{\max}$ is always positive, else A is identically zero and M is a hyperplane, contradicting the compactness assumption. So we can divide (3.15) by $|A|^4_{\max}$, which gives

$$-\partial_t |A|^{-2}_{\max} \leq 2. \tag{3.16}$$

For any times t, s with $t \leq s < T$, integrating (3.16) over $[t, s]$ gives

$$|A(\cdot, t)|_{\max}^{-2} - |A(\cdot, s)|_{\max}^{-2} \leq 2(s-t).$$

Sending $s \to T$, the second term on the left vanishes by Theorem 3.14 and we are left with

$$|A(\cdot, t)|_{\max}^{-2} \leq 2(T-t),$$

which is the result. □

3.3 Blowup limits for singularity analysis

In this section, we carry out a blowup procedure to facilitate analysis of the singular set $\mathcal{S}$ of an MCF. We will magnify around a point $\widehat{p} \in \mathcal{S}$ while approaching the singular time, and extract a limit hypersurface which models the singularity forming at $\widehat{p}$. We need to address two key issues:

- We need a suitable notion of convergence of hypersurfaces which captures the geometric nature of the problem. Moreover, this should admit a compactness (i.e. Arzelà–Ascoli type) theorem to guarantee the existence of a limit hypersurface under mild conditions.
- The rescaling should be performed in such a way that the conditions of said compactness theorem are satisfied. In particular, we must rid the curvature blowup from Theorem 3.14; doing this necessitates an ad hoc assumption, the *Type I hypothesis*. In §3.3.3, we outline how one removes this troubling assumption to get convergence more generally.

3.3.1 A compactness theorem for submanifolds

This subsection is based on [ACGL20, §11]. See also [AH11, §8] and [Bak] for more details on convergence of Riemannian manifolds and compactness theorems.

The appropriate notion of convergence for MCF blowup analysis is *smooth convergence on compact subsets* for a sequence of smooth immersions. Recall that an *exhaustion* of a manifold $\widetilde{M}$ is a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open sets in $\widetilde{M}$ such that $U_i \subset\subset U_{i+1}$ for each k and $\widetilde{M} = \bigcup_{k\in\mathbb{N}} U_i$.

Definition 3.16. A sequence of immersions $\phi_i \in C^\infty(M, \mathbb{R}^{n+1})$ of a manifold M converges to a limit immersion $\phi \in C^\infty(\widetilde{M}, \mathbb{R}^{n+1})$ of a manifold $\widetilde{M}$ *smoothly on compact subsets of* $\mathbb{R}^{n+1}$ if two conditions hold:

(1) There is an exhaustion $\{U_i\}_{i\in\mathbb{N}}$ of $\widetilde{M}$ and a sequence of embeddings $\iota_i : U_i \to M$ such that for each compact subset $K \subset \widetilde{M}$ and each $k \in \mathbb{N}_0$, the sequence of immersions $\iota_i^*\phi_i|_K : U_i \cap K \to \mathbb{R}^{n+1}$ converges in $C^k(K, \mathbb{R}^{n+1})$ to $\phi|_K : K \to \mathbb{R}^{n+1}$.

(2) For each $R < \infty$, there exists $i_R \in \mathbb{N}$ such that $\iota_i^*\phi_i(U_i) \cap \overline{B}_R = \phi_i(M) \cap B_R$ for all $i \geq i_R$, where B_R is the ball of radius R in $\mathbb{R}^{n+1}$ centred at 0.

With this definition, M and $\widetilde{M}$ can be topologically distinct. To illustrate why this is useful for geometric problems, consider a sequence of embedded circles of increasing radius,

$$\phi_k : S^1 \to \mathbb{C} \cong \mathbb{R}^2, \quad \phi_k(e^{i\theta}) = k \cdot (e^{i\theta} + i),$$

as in Figure 3.2. It can be shown that ϕ_k converges to the inclusion $\iota : \mathbb{R} \hookrightarrow \mathbb{C}$ in the sense of Definition 3.16, as it intuitively should. On the contrary, we cannot get a limit immersion in the sense of functions; for example, ϕ_k does not converge to an immersion in $C^\infty(S^1, \mathbb{C})$ since S^1 cannot immerse onto the entire horizontal axis.

This notion of convergence also admits a compactness theorem which guarantees subconvergence to a limiting immersion $\phi : \widetilde{M} \to \mathbb{R}^{n+1}$.

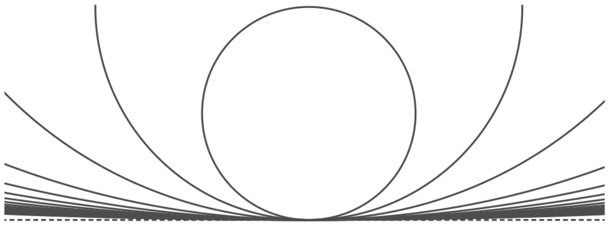

Figure 3.2: Circles of increasing radius converge to the horizontal axis in the sense of Definition 3.16, but not in $C^\infty(S^1, \mathbb{C})$. Adapted from [ACGL20].

Theorem 3.17. *Let $\phi_i : M \to \mathbb{R}^{n+1}$ be a sequence of smooth immersions of a smooth, connected, compact manifold M, and suppose the following hold.*

(1) There is a sequence of points $x_i \in M$ and a constant $A < \infty$ such that $\phi_i(x_i) \in B_A$ for all i.

(2) For each $m \in \mathbb{N}_0$, there exists a constant $C_m < \infty$ such that

$$\max_{p \in M} |\nabla_i^m A_i(p)|_{g_i} \leq C_m$$

for all i, where A_i, ∇_i and g_i are the second fundamental form, Levi-Civita connection and induced metric on M via ϕ_i respectively.

(3) For every $R < \infty$ there exists $C_R < \infty$ such that

$$\mathcal{H}^n(B_R \cap \phi_i(M)) \leq C_R$$

for all i.

Then there exist a smooth n-manifold $\widetilde{M}$, a smooth proper immersion $\phi : \widetilde{M} \to \mathbb{R}^{n+1}$, and a subsequence of $\{\phi_i\}_{i \in \mathbb{N}}$ which converges smoothly on compact subsets of $\mathbb{R}^{n+1}$ to ϕ.

3.3.2 The rescaled flow and convergence

For the remainder of this chapter, we assume M is compact. Given an initial immersion φ_{t_0}, the MCF $\varphi : M \times [t_0, T) \to \mathbb{R}^{n+1}$ is uniquely defined up to time $T < \infty$ by Theorem 3.5 and Corollary 3.8.

Proposition 3.15 gives a curvature blowup rate of at least $\frac{1}{\sqrt{2(T-t)}}$. Following Huisken [Hui90], we shall assume this bound is tight in that there exists $K < \infty$ such that

$$\frac{1}{\sqrt{2(T-t)}} \leq \max_{p \in M} |A(p,t)| \leq \frac{K}{\sqrt{2(T-t)}}. \tag{3.17}$$

This is called the *Type I hypothesis*, and we say that the MCF is of *Type I* (or develops a *Type I singularity*). This is an ad hoc assumption whose sole purpose to make (i) of Theorem 3.17 hold when invoked. Accordingly, a *special singular point* is a point $p \in M$ such that for some fixed $\delta > 0$ and sequence of times $t_i \to T$ we have

$$|A(p, t_i)| \geq \frac{\delta}{\sqrt{2(T - t_i)}}.$$

Remark 3.18. The Type I hypothesis does not immediately imply the existence of a special singular point. On the other hand, it implies the existence of a *singular point*, which is a point $p \in M$ such that there is a sequence of points $p_i \to p$ in M and times $t_i \to T$ with

$$|A(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T - t_i)}},$$

for some fixed $\delta > 0$. However, both notions of singular point actually coincide for MCF satisfying the Type I hypothesis. This result is due to Stone [Sto94] and Le and Sesum [LS11].

Remark 3.19. The caveat is that in most cases, we cannot tell if an MCF is of Type I just by looking at the initial hypersurface. We therefore endeavour to remove the Type I hypothesis; this is discussed in the next subsection, but we retain it for now to simplify the convergence proof.

Using the algebraic inequality $|H|^2 \leq n|A|^2$, a first consequence of the Type I hypothesis is that for any $p \in M$ and $t_0 \leq t \leq s < T$,

$$|\varphi(p, s) - \varphi(p, t)| = \left| \int_t^s \frac{\partial \varphi(p, \xi)}{\partial t} \, d\xi \right| \leq \int_t^s |H(p, \xi)| \, d\xi \leq \int_t^s \frac{K\sqrt{n}}{\sqrt{2(T - \xi)}} \, d\xi \leq K\sqrt{2n(T - t)}. \tag{3.18}$$

This is made arbitrarily small independently of p by bringing t close to T. Thus $\varphi(\cdot, t)$ converges uniformly as $t \to T$ to a function $\varphi_T : M \to \mathbb{R}^{n+1}$, which is continuous since each of the $\varphi(\cdot, t)$'s are. Using this, we define the singular set $\mathcal{S}$ for a Type I MCF by

$$\mathcal{S} = \{\varphi_T(p) \mid p \text{ is a (special) singular point}\}.$$

As a shorthand, we will write $\widehat{p} = \varphi_T(p)$.

Choosing some $\widehat{p}$, we will rescale the MCF around $\widehat{p}$, obtaining a new flow called the *rescaled mean curvature flow* (RMCF). Generally we are interested in doing this for $\widehat{p} \in \mathcal{S}$ to examine the singularity forming there. In the following definition of RMCF, the rescaling factor is chosen to offset the curvature blowup rate of Proposition 3.15 when $\widehat{p} \in \mathcal{S}$. Time is also rescaled to give a new time parameter s taking values up to infinity.

Definition 3.20. The *rescaled mean curvature flow* (RMCF) of a compact MCF $\varphi : M \times [t_0, T) \to \mathbb{R}^{n+1}$ around $\widehat{p}$ is the flow $\widetilde{\varphi}$ defined by[1]

$$\widetilde{\varphi}(q, s) = \frac{\varphi(q, t(s)) - \widehat{p}}{\sqrt{T - t(s)}}, \quad s = s(t) = -\log(T - t),$$

for $(q, s) \in M \times [-\log(T - t_0), \infty)$.

[1]Another convention, following Huisken [Hui90], is to have an extra $\sqrt{2}$ factor in the denominator.

Lemma 3.21. *The RMCF $\widetilde{\varphi}$ (around a given $\widehat{p}$) has normal speed*

$$\left\langle \frac{\partial \widetilde{\varphi}(q,s)}{\partial s}, \widetilde{\mathbf{n}} \right\rangle = -\widetilde{H}(q,s) + \frac{\langle \widetilde{\varphi}(q,s), \widetilde{\mathbf{n}} \rangle}{2}, \tag{3.19}$$

where $\widetilde{H}$ and $\widetilde{\mathbf{n}}$ are the mean curvature and unit normal on M induced by $\widetilde{\varphi}$, respectively. If $\widetilde{\varphi}$ immerses M as a shrinker at every time, then the hypersurfaces $\widetilde{\varphi}(M,s)$ are stationary.

Proof. Since $\frac{ds}{dt} = \frac{1}{T-t}$, we compute

$$\begin{aligned}\frac{\partial \widetilde{\varphi}(q,s)}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1} \frac{\partial}{\partial t} \left(\frac{\varphi(q,t) - \widehat{p}}{\sqrt{T-t}} \right) \\ &= \sqrt{T-t} \cdot \frac{\partial \varphi(q,t)}{\partial t} + \frac{1}{2} \left(\frac{\varphi(q,t) - \widehat{p}}{\sqrt{T-t}} \right) \\ &= -\sqrt{T-t} \cdot H(q,t)\mathbf{n}(q,t) + \frac{1}{2}\widetilde{\varphi}(q,s).\end{aligned}$$

The first term is $-\widetilde{H}(q,s)\widetilde{\mathbf{n}}(q,s)$ by the definition of RMCF. Taking inner products with $\widetilde{\mathbf{n}}$ gives (3.19). If $\widetilde{\varphi}$ immerses a shrinker at each time, then the right-hand side of (3.19) is zero by Theorem 3.9. Hence $\frac{\partial \widetilde{\varphi}}{\partial s}$ is always tangential, so the hypersurface in $\mathbb{R}^{n+1}$ does not change. □

The next few lemmas showcase some important properties that the RMCF of a Type I MCF has. Theorem 3.27 will use these to extract a limit hypersurface as the singularity model at $\widehat{p}$.

Lemma 3.22. *Let φ be a Type I MCF. Let $\widetilde{A}_s$, $\widetilde{\nabla}_s$ and $\widetilde{g}_s$ be the second fundamental form, Levi-Civita connection and induced metric on M via $\widetilde{\varphi}(\cdot,s)$ respectively, where $\widetilde{\varphi}$ is the RMCF (around a given $\widehat{p}$). For each $m \in \mathbb{N}_0$, there exists $C_m = C_m(m,n,K) < \infty$ such that for all s,*

$$\max_{p \in M} |\widetilde{\nabla}_s^m \widetilde{A}_s(p)|_{\widetilde{g}_s} < C_m.$$

Proof. The following calculations are at an arbitary $p \in M$, and we suppress the time parameter s in notation. Using $|\widetilde{\nabla}^m \widetilde{A}|^2_{\widetilde{g}} = (T-t)^{m+1} |\nabla^m A|^2_g$, (3.9) and the inequality $ab \leq \frac{1}{2}(a^2+b^2)$,

$$\begin{aligned}\frac{\partial}{\partial s}|\widetilde{\nabla}^m \widetilde{A}|^2_{\widetilde{g}} &= \left(\frac{\partial s}{\partial t}\right)^{-1} \frac{\partial}{\partial t}\left((T-t)^{m+1}|\nabla^m A|^2_g\right) \\ &= (T-t)\left\{-(m+1)(T-t)^m |\nabla^m A|^2_g + (T-t)^{m+1}\frac{\partial}{\partial t}|\nabla^m A|^2_g\right\} \\ &\leq -(m+1)|\widetilde{\nabla}^m \widetilde{A}|^2_{\widetilde{g}} + (T-t)^{m+2}\Big\{\Delta|\nabla^m A|^2_g - 2|\nabla^{m+1}A|^2_g \\ &\qquad + C(m,n) \sum_{p+q+r=m|p,q,r\in\mathbb{N}} |\nabla^p A|_g |\nabla^q A|_g |\nabla^r A|_g |\nabla^m A|_g \Big\} \qquad (3.20) \\ &= \widetilde{\Delta}|\widetilde{\nabla}^m \widetilde{A}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{m+1}\widetilde{A}|^2_{\widetilde{g}} + C(m,n) \sum_{p+q+r=m|p,q,r\in\mathbb{N}} |\widetilde{\nabla}^p \widetilde{A}|_{\widetilde{g}} |\widetilde{\nabla}^q \widetilde{A}|_{\widetilde{g}} |\widetilde{\nabla}^r \widetilde{A}|_{\widetilde{g}} |\widetilde{\nabla}^m \widetilde{A}|_{\widetilde{g}} \\ &\leq \widetilde{\Delta}|\widetilde{\nabla}^m \widetilde{A}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{m+1}\widetilde{A}|^2_{\widetilde{g}} + C(m,n) \sum_{p+q+r=m|p,q,r\in\mathbb{N}} \left\{|\widetilde{\nabla}^m \widetilde{A}|^2_{\widetilde{g}} + |\widetilde{\nabla}^p \widetilde{A}|^2_{\widetilde{g}} |\widetilde{\nabla}^q \widetilde{A}|^2_{\widetilde{g}} |\widetilde{\nabla}^r \widetilde{A}|^2_{\widetilde{g}}\right\}.\end{aligned}$$

Proceed by induction, where the base case is satisfied since (3.17) gives $|\widetilde{A}|_{\widetilde{g}} = (T-t)^{1/2}|\widetilde{A}|_g \leq K/\sqrt{2}$. Assume we have uniform bounds $|\widetilde{\nabla}^k \widetilde{A}|_{\widetilde{g}} \leq C_k$ for $k = 0, \ldots, m-1$ where $C_k =$

$C_k(k,n,K)$. Then (3.20) becomes

$$\frac{\partial}{\partial s}|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 \le \widetilde{\Delta}|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 - 2|\widetilde{\nabla}^{m+1}\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + D_m, \tag{3.21}$$

where B_m and D_m depend only on m,n and $C_0,\dots,C_{m-1}$, thus on m,n,K. By a computation using (3.21) and the same inequality with m replaced by $m-1$, we have (see [Man11, Proposition 3.2.9])

$$\frac{\partial}{\partial s}(|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}}^2) \le \widetilde{\Delta}(|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}}^2) - B_m(|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}}^2) + E_m,$$

where $E_m = E_m(m,n,K)$. By the maximum principle (Theorem 2.5), we get that

$$\frac{\partial}{\partial s}\max_M(|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}}^2) \le -B_m\max_M(|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}}^2) + E_m,$$

and Lemma 2.7 now gives an estimate for $|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}}^2$ with exponentially decaying error in time. Thus $|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}}^2 + B_m|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}}^2$ is uniformly bounded in space and time by a constant depending on m,n and K. But $|\widetilde{\nabla}^{m-1}\widetilde{A}|_{\widetilde{g}} \le C_{m-1}$ also, so $|\widetilde{\nabla}^m\widetilde{A}|_{\widetilde{g}} \le C_m$. □

Lemmas 3.24 and 3.25 below will stem from Huisken's celebrated monotonicity formula:

Lemma 3.23 (Huisken's monotonicity formula, [Hui90])**.** *Let φ be an (unscaled) mean curvature flow. For $x_0 \in \mathbb{R}^{n+1}$ and $\tau \in \mathbb{R}$, consider a reversed heat kernel $\rho_{x_0,\tau} : \mathbb{R}^{n+1} \times (-\infty,\tau) \to \mathbb{R}$,*

$$\rho_{x_0,\tau}(x,t) = (4\pi(\tau-t))^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4(\tau-t)}}.$$

Then for all $t \in [t_0, \min\{\tau,T\})$ we have

$$\frac{d}{dt}\int_M \rho_{x_0,\tau}(x,t)\,d\mu_t = -\int_M \left(H - \frac{\langle x-x_0,\mathbf{n}\rangle}{2(\tau-t)}\right)^2 \rho_{x_0,\tau}(x,t)\,d\mu_t \le 0,$$

with equality if and only if φ is a self-shrinking MCF about the spacetime point (x_0,τ).

Proof. Reparametrising if needed, we may assume $\partial_t\varphi = -H\mathbf{n}$. For any smooth function $f : \mathbb{R}^{n+1} \times I \to \mathbb{R}$, we have (by the chain rule and Proposition 2.1)

$$\frac{d}{dt}\int_M f\,d\mu_t = \frac{d}{dt}\int_M f(\varphi(p,t),t)\,d\mu_t = \int_M (-H\langle\nabla f,\mathbf{n}\rangle + \partial_t f - H^2 f)\,d\mu_t.$$

The monotonicity formula follows from direct calculation, using $f = \rho_{x_0,\tau}$. For the last part of the lemma, we must show that every self-shrinking MCF about (x_0,τ) has $H = \frac{\langle x-x_0,\mathbf{n}\rangle}{2(\tau-t)}$ everywhere. If $(x_0,\tau) = (0,0)$ and we look at the $t=-1$ timeslice, this is exactly what Theorem 3.9 shows. For the general case with arbitrary (x_0,τ) and t, one can obtain the result by mimicking exactly the proof of Theorem 3.9. □

We will rescale this to get a corresponding monotonicity formula for RMCF. We denote generic points on the hypersurfaces $\widetilde{\varphi}(M,s)$ by y, as exemplified in the integrals of the next lemma.

Lemma 3.24. *The RMCF $\widetilde{\varphi}$ satisfies a rescaled version of Huisken's monotonicity formula,*

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_s = -\int_M \left(\widetilde{H} - \frac{\langle y, \widetilde{\mathbf{n}}\rangle}{2}\right)^2 e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_s \le 0. \tag{3.22}$$

As a consequence,

$$\int_{-\log(T-t_0)}^{\infty}\int_M \left(\widetilde{H} - \frac{\langle y, \widetilde{\mathbf{n}}\rangle}{2}\right)^2 e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_s\,ds \le \int_M e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_{-\log(T-t_0)} < \infty. \tag{3.23}$$

Proof. The derivation of (3.22) is a direct computation using Lemma 3.23 and the definition of RMCF, so we omit it. Since the equality in (3.22) gives that the integral $\int_M e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_s$ is positive and nonincreasing, it has a nonnegative limit as $s \to \infty$. Hence, multiplying (3.22) by -1 and integrating from $s = -\log(T - t_0)$ to $s = \infty$ gives

$$\begin{aligned}\int_{-\log(T-t_0)}^{\infty}\int_M \left(\widetilde{H} - \frac{\langle y, \widetilde{\mathbf{n}}\rangle}{2}\right)^2 e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_s\,ds &= \int_M e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_{-\log(T-t_0)} - \int_M e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_\infty \\ &\le \int_M e^{-\frac{|y|^2}{4}}\,d\widetilde{\mu}_{-\log(T-t_0)} = \int_M (T-t_0)^{-\frac{n}{2}} e^{-\frac{|x-\widetilde{p}|^2}{4(T-t_0)}}\,d\mu_{t_0} \\ &\le (T-t_0)^{-\frac{n}{2}}\mathcal{H}^n(\varphi_{t_0}(M)) < \infty,\end{aligned}$$

where the equality in the second line scales back to the original MCF. This proves (3.23). □

The art of monotonicity formulae is not so much in their proofs as opposed to deciding what quantity to show monotonicity for. In our case, Huisken's monotonicity formula makes the enlightening link between blowup limits of (R)MCF and shrinkers. Before seeing this, let us state an important local area (or volume) bound for RMCF that also uses monotonicity.

Lemma 3.25 ([CM12])**.** *Let $s' > s_0 = -\log(T - t_0)$. For all $s \ge s'$ and $x_0 \in \mathbb{R}^{n+1}$, the rescaled hypersurfaces $\widetilde{\varphi}_s(M)$ satisfy the polynomial volume bound*

$$\mathcal{H}^n(B_R(x_0) \cap \widetilde{\varphi}_s(M)) \le VR^n$$

for all $R > 0$, where V depends on s' and $\mathcal{H}^n(\widetilde{\varphi}_{s_0}(M))$, and in particular not on s nor x_0.

Proof. We prove it for the *unscaled* hypersurfaces $\varphi_t(M)$. That is, whenever $t \ge t' > t_0$ and $x_0 \in \mathbb{R}^{n+1}$, there exists $V = V(\mathcal{H}^n(\varphi_{t_0}(M)), t')$ such that $\mathcal{H}^n(B_R(x_0) \cap \varphi_t(M)) \le VR^n$ for all $R > 0$. The conclusion for the rescaled hypersurfaces follows by scaling.

Let $\tau > t \ge t' > t_0$, where τ will be chosen later. We can bound

$$\begin{aligned}(4\pi(\tau-t))^{-\frac{n}{2}} e^{-\frac{1}{4}}\mathcal{H}^n(B_{\sqrt{\tau-t}}(x_0) \cap \varphi_t(M)) &\le (4\pi(\tau-t))^{-\frac{n}{2}}\int_{B_{\sqrt{\tau-t}}(x_0)\cap\varphi_t(M)} e^{-\frac{|x-x_0|^2}{4(\tau-t)}} \\ &\le \int_{\varphi_t(M)} \rho_{x_0,\tau}(x,t) \le \int_M \rho_{x_0,\tau}(x,t_0)\,d\mu_{t_0} = (4\pi(\tau-t_0))^{-\frac{n}{2}}\int_{\varphi_{t_0}(M)} e^{-\frac{|x-x_0|^2}{4(\tau-t_0)}} \\ &\le (4\pi(\tau-t_0))^{-\frac{n}{2}}\mathcal{H}^n(\varphi_{t_0}(M)) \le (4\pi(t'-t_0))^{-\frac{n}{2}}\mathcal{H}^n(\varphi_{t_0}(M)),\end{aligned}$$

where the second inequality in the second line is Lemma 3.23. Rearranging gives

$$\mathcal{H}^n(B_{\sqrt{\tau-t}}(x_0) \cap \varphi_t(M)) \le \left(\frac{\tau-t}{t'-t_0}\right)^{\frac{n}{2}} e^{\frac{1}{4}}\mathcal{H}^n(\varphi_{t_0}(M)),$$

so choosing $\tau = t + R^2$ gives the claim with $V = (t' - t_0)^{-\frac{n}{2}} e^{\frac{1}{4}} \mathcal{H}^n(\varphi_{t_0}(M))$. □

Definition 3.26. An immersed hypersurface $\psi : M \to \mathbb{R}^{n+1}$ has *polynomial volume growth* if there is a constant V such that for all $R > 0$ and $x_0 \in \mathbb{R}^{n+1}$, we have $\mathcal{H}^n(B_R(x_0) \cap \psi(M)) \leq V R^n$.

We are ready to state the main theorem of this section, which says that a limit hypersurface not only exists, but satisfies the shrinker equation and has polynomial volume growth.

Theorem 3.27 ([Hui90]). *Let φ be a compact Type I MCF, and $\widetilde{\varphi}$ its RMCF around a point $\widehat{p}$. For every sequence of times $\{s_i\}_{i \in \mathbb{N}}$ with $s_i \to \infty$, there is a subsequence of $\{\widetilde{\varphi}(\cdot, s_i)\}$ which converges smoothly on compact subsets of $\mathbb{R}^{n+1}$ to a proper immersion $\widetilde{\varphi}_\infty : \widetilde{M}_\infty \to \mathbb{R}^{n+1}$ of a hypersurface.*

The limit manifold $\widetilde{M}_\infty$ satisfies $\widetilde{H}_\infty = \frac{\langle y, \widetilde{\mathbf{n}}_\infty \rangle}{2}$ at all points, thus is a shrinker by Theorem 3.9 and Remark 3.10. Moreover, $\widetilde{M}_\infty$ has polynomial volume growth, and for every $m \in \mathbb{N}_0$ there exists $C_m < \infty$ such that $|\widetilde{\nabla}^m \widetilde{A}|_{\widetilde{g}} \leq C_m$ on $\widetilde{M}_\infty$.

If the initial hypersurface of φ was mean convex ($H \geq 0$), then so is $\widetilde{M}_\infty$.

If the initial hypersurface of φ was embedded, then so is $\widetilde{M}_\infty$.

Proof sketch. The first paragraph of the theorem follows by Theorem 3.17 as long as the family $\{\widetilde{\varphi}(\cdot, s_i)\}_{i \in \mathbb{N}}$ satisfies conditions (1)-(3) listed there. First divide both sides of (3.18) by $\sqrt{T - t}$ and send $t \nearrow T$ to get $|\widetilde{\varphi}(p, s_i)| = \left| \frac{\varphi(p, t(s_i)) - \widehat{p}}{\sqrt{T - t(s_i)}} \right| \leq K\sqrt{2n}$. Then (1) is fulfilled using $x_i \equiv p$ and $R = K\sqrt{2n}$. The condition (2) is the result of Lemma 3.22, while (3) is Lemma 3.25.

Write $\widetilde{\Sigma}_s = \widetilde{\varphi}(M, s)$ and $\widetilde{\Sigma}_\infty = \widetilde{\varphi}_\infty(\widetilde{M}_\infty)$. For the second set of claims, we need to use that the measures $\mathcal{H}^n \llcorner \widetilde{\Sigma}_{s_i}$ weakly-* converge to $\mathcal{H}^n \llcorner \widetilde{\Sigma}_\infty$, and that limits of integrals over $\widetilde{\Sigma}_{s_i}$ as $i \to \infty$ are integrals over $\widetilde{\Sigma}_\infty$. See [Sto94] or [Man11] for details. Then,

$$\int_{\widetilde{\Sigma}_\infty} \left(\widetilde{H}_\infty - \frac{\langle y, \widetilde{\mathbf{n}}_\infty \rangle}{2} \right)^2 e^{-\frac{|y|^2}{4}} = \lim_{i \to \infty} \int_M \left(\widetilde{H} - \frac{\langle y, \widetilde{\mathbf{n}} \rangle}{2} \right)^2 e^{-\frac{|y|^2}{4}} d\widetilde{\mu}_{s_i} = 0,$$

where the second equality uses (3.23). So $\widetilde{H}_\infty = \frac{\langle y, \widetilde{\mathbf{n}}_\infty \rangle}{2}$ everywhere on $\widetilde{\Sigma}_\infty$. Lemmas 3.22 and 3.25 give uniform curvature bounds and polynomial volume growth on $\widetilde{\Sigma}_{s_i}$ for all i. These are all local properties, so the smooth convergence forces $\widetilde{\Sigma}_\infty$ (equivalently $\widetilde{M}_\infty$) to inherit them. If the initial hypersurface of φ was mean convex, then Proposition 3.13 gives that φ, and hence $\widetilde{\varphi}$, is mean convex at all times. By the smooth convergence, $\widetilde{\Sigma}_\infty$ is also mean convex.

For details on the final part of the theorem, see [Man11, Proposition 3.2.10]. □

Definition 3.28. A *tangent flow* of an MCF (with respect to some RMCF) is a limit hypersurface obtained the above way, that is by passing to a subsequence of times $s_i \to \infty$.

3.3.3 Convergence without the type I hypothesis

Theorem 3.27 says that for a Type I MCF, tangent flows exist and are shrinkers. However, the Type I hypothesis is largely unverifiable, so it would be awkward to base an entire theory on this. White [Whi94] and Ilmanen [Ilm95] rectified this by generalising Theorem 3.27 as follows. If

φ is any MCF (not necessarily Type I) and $\widetilde{\varphi}$ is its RMCF as in Definition 3.20, then we can still extract tangent flows *in the sense of varifolds*, which will be shrinkers in a weak sense.

Varifolds are measure-theoretic generalisations of smooth manifolds, defined as Radon measures on Euclidean space satisfying certain rectifiability conditions (see [Sim83b]). One can study a version of MCF for varifolds, called *Brakke flow* due to Brakke's original monograph [Bra78]. Every MCF is a Brakke flow, and the concepts of shrinkers, convergence on compact subsets, the compactness theorem (Theorem 3.17) and Huisken's monotonicity formula all generalise to Brakke flows. Using these, White and Ilmanen's generalisation of Theorem 3.27 says that rescaling a Brakke flow analogously to RMCF gives subconvergence to varifolds that weakly satisfy the shrinker equation $H = \frac{\langle x, \mathbf{n} \rangle}{2}$. In line with the literature, *tangent flows* will refer to these Brakke limits rather than the stronger smooth limits discussed earlier.

One may ask whether the tangent flow is actually a smooth hypersurface, and if so, whether the convergence is smooth on compact subsets. White [Whi03] showed that for a smooth, embedded, mean convex MCF, every tangent flow is smooth with unit multiplicity. By Brakke's regularity theorem [Bra78], this implies the convergence is smooth. Thus, in this special case, the convergence is no different to that of Theorem 3.27, except the Type I hypothesis can be dropped.

Theorem 3.29. *If φ is a compact, embedded, mean convex MCF, and Σ is a tangent flow of φ, then Σ is a unit multiplicity smooth, embedded, mean convex shrinker with polynomial volume growth and uniform bounds on A and all covariant derivatives. Convergence to Σ is smooth on compact subsets of $\mathbb{R}^{n+1}$.*

We also need to mention the *compactness theorem for MCFs* (see, e.g. [ACGL20, Theorem 11.12]): for every sequence of 'nice enough' MCFs $\{\varphi_i : M_i \times I_i \to \mathbb{R}^{n+1}\}_{i\in\mathbb{N}}$, there is a subsequence converging to another MCF on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$. A *compactness theorem for RMCFs* holds by rescaling. The 'nice' conditions to apply these are similar to Theorem 3.17, but we leave an exact statement to the reference above; we note only that the conditions will be met whenever we want to use the theorems. Similar compactness theorems hold for Brakke flows and rescaled Brakke flows using a measure-theoretic notion of convergence [Bra78]. We also remark that Theorem 3.27 (and its weak generalisation) admits an equivalent formulation in terms of convergence of a sequence of scaled MCFs (resp. Brakke flows). See, e.g. [ACGL20, Theorem 11.26] for the smooth case. This is why the limits are called tangent *flows*.

3.4 The uniqueness of tangent flows problem

What do singularities in MCF look like? The RMCF was conceived as a means to answer this question, and so far this has paid off: by Theorem 3.27 and its weak generalisation they weakly resemble shrinkers, at least when the initial hypersurface is compact.

The first step in classifying singularities is therefore to classify the shrinkers. We will concentrate exclusively on the smooth case, henceforth assuming all hypersurfaces are smooth. In dimension $n = 1$, a full classification of shrinkers is known: see [AL86, EW87, Hal12]. Less is known for $n \geq 2$, but in all dimensions the only embedded, mean convex shrinkers with polynomial volume growth are hyperplanes, cylinders and spheres [Hui90, CM12]. Specifically, they are rotations of

$S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$ for $k \in \{0, \dots, n\}$.[2] This will be proved in §4. It follows by Theorem 3.29 that every tangent flow arising from a compact, embedded, mean convex MCF is of this type.

To make this classification useful for singularity analysis, we need to address another question. Recall that tangent flows are limits of an RMCF along a sequence of times $s_i \to \infty$. It could well be that under different sequences of times, the extracted tangent flows look completely different. Whether or not this happens is the *uniqueness of tangent flows* problem.

Question 3.30. *Are tangent flows unique? In other words, does a tangent flow obtained from an RMCF depend on the sequence of times $s_i \to \infty$ along which the limit is extracted?*

This is a highly fundamental question. Firstly, uniqueness of tangent flows allows us to equate singularity classification with shrinker classification. Singularities can be classified by the geometric type of a tangent flow of its RMCF ('spherical' or 'cylindrical' for instance), but we need uniqueness of tangent flows to guarantee this classification is even well-defined. Secondly, and perhaps more importantly, uniqueness tells us a great deal about the structure and regularity of the singular set $\mathcal{S}$. This could form the basis for sophisticated developments like MCF with surgery which, up until now, have only enjoyed limited success. See §7 for a discussion.

The first answers to the uniqueness question go back to Huisken [Hui84] and Gage–Hamilton–Grayson [GH86, Gra87]. Together, they showed that every tangent flow at a singular point of a convex, embedded MCF is the sphere $S^n_{\sqrt{2n}}$ (convexity is not needed for $n = 1$). Much later, uniqueness was proved to hold for compact tangent flows [Sch14] and cylindrical tangent flows [CM15]. These results are detailed in §5 and §6, respectively. Among recent achievements are the uniqueness of asymptotically conical tangent flows [CS21] and of cylindrical tangent flows in high codimension [CM19b], which we leave to a discussion in the final chapter. On the other hand, by the Brakke–White regularity theorem [Whi05], hyperplanes arise as tangent flows if and only if the flow is rescaled around a nonsingular point. These are indeed unique and while not hard to prove, a technical setup is still required (see [Man11, Theorem 3.2.22] and discussion thereafter). We will skip this in favour of proving uniqueness for nontrivial blowups.

We can now give a precise version of Theorem 1.1, the central theorem of this thesis.

Theorem 3.31. *Let φ be a mean curvature flow of compact, embedded, mean convex hypersurfaces in $\mathbb{R}^{n+1}$, and let $\widetilde{\varphi}$ be an RMCF of it. There exists $k \in \{0, \dots, n\}$ and a rotation of $\mathbb{R}^{n+1}$ such that every tangent flow of $\widetilde{\varphi}$ is that rotation of $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$. Along every sequence of times $s_i \to \infty$, convergence of $\widetilde{\varphi}(\cdot, s_i)$ to the tangent flow is smooth on compact subsets of $\mathbb{R}^{n+1}$.*

Proof. By the discussion in this section, all tangent flows of such an MCF are hyperplanes, cylinders and spheres, and these are all unique. Except for uniqueness of planar tangent flows, we will prove these assertions in the next three chapters (Corollary 4.2, Theorem 5.1 and Theorem 6.1). □

[2] The cases $k = 0$ and $k = n$ are understood to be hyperplanes and spheres respectively.

Chapter 4

Classification of Mean Convex Shrinkers

The goal of this chapter is to prove the following classification result.

Theorem 4.1 ([Hui90, CM12]). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth embedded shrinker with polynomial volume growth and $H \geq 0$ everywhere. Then Σ is a rotation of $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$ for some $k \in \{0, \dots, n\}$.*

Combining this with Theorem 3.29 immediately gives the following corollary.

Corollary 4.2. *Every tangent flow of a compact, embedded, mean convex MCF is a rotation of $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$ with unit multiplicity.*

In [Hui90], the same classification was achieved assuming $|A|$ is bounded on Σ. While Theorem 3.29 does indeed establish this bound, we present the stronger result since it uses machinery that will resurface in later chapters. We will first introduce this machinery in §4.1, before turning to the proof of Theorem 4.1 in §4.2. The exposition in §4.1 is our own, while the statements and proofs in §4.2 are essentially those of [CM12] with some reorganisation and added detail.

4.1 Some machinery: the $\mathcal{F}$-functional and weighted L^p spaces

Let $\Sigma \subset \mathbb{R}^{n+1}$ be an embedded hypersurface with polynomial volume growth. Adapting the setup of §2.1 to our present discussion, the inclusion $\iota : \Sigma \hookrightarrow \mathbb{R}^{n+1}$ induces a metric $g = \iota^* g^{\mathrm{Euc}}$ on Σ, where g^{Euc} is the Euclidean metric on $\mathbb{R}^{n+1}$. Using this, we get a measure $\mu = \sqrt{\det(g_{ij})}\mathcal{L}^n$ on Σ, where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$, and the coefficients g_{ij} are with respect to a local chart for Σ. Now suppose we repeat this process, replacing g^{Euc} by the *Gaussian metric* on $\mathbb{R}^{n+1}$,

$$g^{\mathrm{Gauss}}(x) = e^{-\frac{|x|^2}{2n}} g^{\mathrm{Euc}}(x), \qquad x \in \mathbb{R}^{n+1},$$

thereby obtaining a new measure ν on Σ. One checks that $\frac{d\nu}{d\mu} = e^{-\frac{|x|^2}{4}}$. Polynomial volume growth leads means Σ has finite weighted volume (or *Gaussian area*):

$$\nu(\Sigma) = \int_\Sigma d\nu = \int_\Sigma e^{-\frac{|x|^2}{4}} d\mu = \sum_{R=1}^{\infty} \int_{(B_R \setminus B_{R-1}) \cap \Sigma} e^{-\frac{|x|^2}{4}} d\mu \leq \sum_{R=1}^{\infty} V R^n e^{-\frac{(R-1)^2}{4}} < \infty. \qquad (4.1)$$

In the MCF literature, the $\mathcal{F}$*-functional* is $(4\pi)^{-\frac{n}{2}}$ times the Gaussian area:

$$\mathcal{F}(\Sigma) = (4\pi)^{-\frac{n}{2}}\nu(\Sigma) = (4\pi)^{-\frac{n}{2}}\int_\Sigma e^{-\frac{|x|^2}{4}},$$

where the last integral is evaluated against $d\mu$. Note that $\mathcal{F}$ is nonincreasing during an RMCF by Huisken's monotonicity formula, Lemma 3.24.

To make $\mathcal{F}$ a genuine functional, we reformulate $\mathcal{F}$ as being defined on sections of the normal bundle of a *fixed* hypersurface Σ, which are identified with real-valued functions on Σ. Since Σ has normal injectivity radius $\delta > 0$,[1] every C^1 function ψ with $\sup_\Sigma |\psi| < \delta$ gives rise to an embedded C^1 hypersurface $\Sigma_\psi = \{x + \psi(x)\mathbf{n}(x) : x \in \Sigma\}$. Thus, we define $\mathcal{F}_\Sigma$ as a genuine functional $B_\delta(0) \cap C^1(\Sigma) \to \mathbb{R}$ by

$$\mathcal{F}_\Sigma(\psi) = \mathcal{F}(\Sigma_\psi) = (4\pi)^{-\frac{n}{2}}\int_{\Sigma_\psi} e^{-\frac{|x|^2}{4}}. \tag{4.2}$$

Small C^1 norm ensures this is finite (see §B.2). The next theorem extends the characterisation of shrinkers in Theorem 3.9 when assuming polynomial volume growth.

Theorem 4.3. *Embedded shrinkers in $\mathbb{R}^{n+1}$ with polynomial volume growth are precisely the embedded minimal hypersurfaces in $\mathbb{R}^{n+1}$ with respect to the Gaussian area functional.*

Proof. Let Σ be an embedded shrinker with polynomial volume growth. We show that the zero function is a critical point of the $\mathcal{F}_\Sigma$-functional. Letting $\psi \in C^1(\Sigma)$, we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{F}_\Sigma(\varepsilon\psi) &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(4\pi)^{-\frac{n}{2}}\int_{\Sigma_{\varepsilon\psi}} e^{-\frac{|x|^2}{4}} \\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(4\pi)^{-\frac{n}{2}}\int_\Sigma e^{-\frac{|x+\varepsilon\psi(x)\mathbf{n}(x)|^2}{4}}\, d\mu_{\varepsilon\psi} \\
&= (4\pi)^{-\frac{n}{2}}\int_\Sigma \left(\psi e^{-\frac{|x|^2}{4}} \cdot -\frac{1}{2}\langle x, \mathbf{n}(x)\rangle + H\psi e^{-\frac{|x|^2}{4}}\right) \\
&= (4\pi)^{-\frac{n}{2}}\int_\Sigma \left(H - \frac{\langle x, \mathbf{n}(x)\rangle}{2}\right)\psi e^{-\frac{|x|^2}{4}},
\end{aligned} \tag{4.3}$$

where the third equality uses (2.6) with $X = \psi\mathbf{n}$. This vanishes for all ψ if and only if $H(x) = \frac{\langle x, \mathbf{n}(x)\rangle}{2}$ on Σ, which is precisely when Σ is a shrinker by Theorem 3.9. □

Theorem 4.3 motivates the use of tools from minimal surfaces to study shrinkers, and by extension singularities in mean curvature flow. Huisken's monotonicity formula is already one example; there is a monotonicity formula in minimal surfaces which is used to characterise blowup limits of minimal surfaces as minimal cones (see [CM11]). To go further with this, we need to introduce Gaussian versions of some functional-analytic tools.

In this chapter, we exclusively use the *(Gaussian-)weighted spaces* $L^p(\Sigma)$. For $1 \le p \le \infty$, we define $L^p(\Sigma)$ as the Banach space of measurable functions $u : \Sigma \to \mathbb{R}$ for which the norm

$$\|u\|_{L^p(\Sigma)} = \begin{cases} \left\{\int_\Sigma |u(x)|^p e^{-\frac{|x|^2}{4}}\right\}^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\Sigma} |u(x)| & \text{if } p = \infty, \end{cases}$$

[1] Embedded Euclidean submanifolds have nonzero normal injectivity radius; see [Lee12, Theorem 6.24].

is finite. Like in the unweighted case, $L^2(\Sigma)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{L^2(\Sigma)} = \left\{ \int_\Sigma uve^{-\frac{|x|^2}{4}} \right\}^{1/2}. \tag{4.4}$$

As in §2.2.1, the weighted Sobolev spaces $W^{k,p}$ are defined using these weighted L^p norms. The Hölder spaces $C^{k,\alpha}$ are the same as the unweighted ones since the L^∞ norm has not changed.

An important operator is the *Ornstein–Uhlenbeck operator* $\mathcal{L} : C^2(\Sigma) \to C^0(\Sigma)$, given by

$$\mathcal{L} = \Delta - \frac{1}{2} \langle x, \nabla(\cdot) \rangle_{\mathbb{R}^{n+1}} = e^{\frac{|x|^2}{4}} \operatorname{div} \left(e^{-\frac{|x|^2}{4}} \nabla(\cdot) \right), \tag{4.5}$$

where Δ and ∇ are the Laplacian and gradient on Σ respectively. This is the weighted analogue of the Laplacian in the following sense. Recall that $\Delta = \operatorname{div} \circ \nabla$, and div is minus the formal $L^2(\mu)$-adjoint of ∇ by (2.9). If we let δ be the formal $L^2(\nu)$-adjoint of ∇, then $\mathcal{L} = \delta \circ \nabla$. We will need the following properties of $\mathcal{L}$ in the sequel.

Proposition 4.4 ([CM12]). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be an embedded hypersurface with polynomial volume growth, and let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator defined in* (4.5).

(i) If $u \in W^{1,2}$ is compactly supported and $v \in W^{2,2}$,

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|x|^2}{4}} = -\int_\Sigma \langle \nabla u, \nabla v \rangle \, e^{-\frac{|x|^2}{4}}. \tag{4.6}$$

(ii) If $u \in W^{1,2}$ and $v \in W^{2,2}$, not necessarily compactly supported, and

$$\int_\Sigma (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{|x|^2}{4}} < \infty,$$

then (4.6) *still holds. In particular, this holds if $u, v \in W^{2,2}$, so $\mathcal{L}$ is self-adjoint on $W^{2,2}$.*

(iii) $\mathcal{L}$ has discrete spectrum with finite multiplicity on $W^{2,2}$, and $W^{2,2}$ is spanned by a complete basis of smooth L^2-orthonormal eigenfunctions.

Next, let $\mathcal{M}^\nu_\Sigma : C^2(\Sigma) \to C^0(\Sigma)$ be the $L^2(\nu)$ Euler–Lagrange functional of $\mathcal{F}_\Sigma$, defined by

$$\frac{d}{ds}\bigg|_{s=0} \mathcal{F}_\Sigma(u + sv) = -(4\pi)^{-\frac{n}{2}} \int_\Sigma v\mathcal{M}^\nu_\Sigma(u)e^{-\frac{|x|^2}{4}}. \tag{4.7}$$

We also need the *stability operator* $L : C^2(\Sigma) \to C^0(\Sigma)$, given by

$$L = \Delta + |A|^2 + \frac{1}{2} - \frac{1}{2} \langle x, \nabla(\cdot) \rangle_{\mathbb{R}^{n+1}} = \mathcal{L} + |A|^2 + \frac{1}{2}. \tag{4.8}$$

This is an elliptic operator whose importance stems from the following fact.

Proposition 4.5. *If Σ is an embedded shrinker, then L is the linearisation of $\mathcal{M}^\nu_\Sigma$ at zero. As a consequence, for any $\psi \in C^2(\Sigma)$, one has*

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathcal{F}_\Sigma(\varepsilon\psi) = -(4\pi)^{-\frac{n}{2}} \langle \psi, L\psi \rangle_{L^2(\Sigma)}.$$

This is proved in Appendix B.2. Hence, if L is negative definite, then the shrinker Σ is not only a critical point for the $\mathcal{F}_\Sigma$-functional, but locally minimises $\mathcal{F}_\Sigma$. If this is so, we call Σ a *stable* shrinker. This mirrors the notion of stable minimal hypersurfaces in unweighted space (there the stability operator is $\Delta + |A|^2 + \operatorname{Ric}(\mathbf{n}, \mathbf{n})$; see [CM11]). This plays a central role in Colding and Minicozzi's framework of *generic mean curvature flow* which will be discussed in §7.1.

4.2 Proof of Theorem 4.1

Throughout this section, Σ is an embedded shrinker with polynomial volume growth. We will impose $H \geq 0$ later. We first collect some facts about the stability operator L defined by (4.8).

Lemma 4.6. *On Σ we have $LH = H$ and $LA = A$, where L extends to tensors in the natural way. Furthermore, if $|A|$ does not vanish, then*

$$L|A| = |A| + \frac{|\nabla A|^2 - |\nabla|A||^2}{|A|} \geq |A|. \tag{4.9}$$

Proof. Working in an orthonormal frame $\{e_1, \ldots, e_n\}$ at a point $p \in \Sigma$, the Christoffel symbols vanish at p, so ∇ coincides with the Euclidean directional derivative $\overline{\nabla}$. Also $g_{ij} = \delta_{ij}$ at p. Differentiating the shrinker equation $2H = \langle x, \mathbf{n}\rangle$, we get (at p)

$$2\nabla_j H = \overline{\nabla}_j \langle x, \mathbf{n}\rangle = \langle e_j, \mathbf{n}\rangle + \langle x, -h_{jl}e_l\rangle = -h_{jl}\langle x, e_l\rangle . \tag{4.10}$$

with an implicit sum over $l = 1, \ldots, n$. The second equality uses (2.2) with φ being the inclusion $\iota : \Sigma \hookrightarrow \mathbb{R}^{n+1}$. Differentiating again and keeping in mind that $\overline{\nabla}_i e_l = h_{il}\mathbf{n}$ by (2.2),

$$\begin{aligned} 2\nabla_i\nabla_j H &= -\overline{\nabla}_i(h_{jl}\langle x, e_l\rangle) = -\nabla_i h_{jl}\langle x, e_l\rangle - h_{jl}\langle e_i, e_l\rangle - h_{jl}h_{il}\langle x, \mathbf{n}\rangle \\ &= -\nabla_i h_{jl}\langle x, e_l\rangle - h_{ij} - 2Hh_{jl}h_{il}. \end{aligned}$$

Substituting this into Simons' equation (2.5), we have

$$\Delta h_{ij} = -\nabla_i\nabla_j H - Hh_{il}h_{jl} - |A|^2 h_{ij} = \frac{1}{2}\nabla_i h_{jl}\langle x, e_l\rangle + \frac{1}{2}h_{ij} - |A|^2 h_{ij}. \tag{4.11}$$

Since $\nabla_i h_{jl} = \nabla_l h_{ij}$ by Codazzi's equations (2.4), the above equation reads

$$\Delta A = \frac{1}{2}\langle x, \nabla A\rangle + \frac{1}{2}A - |A|^2 A = -LA + \Delta A + A,$$

so $LA = A$ at p. Since traces commute with covariant derivatives (owing to the metric being parallel, $\nabla g = 0$), taking the negative trace of $LA = A$ gives $LH = H$. To get (4.9), we first recall that $\Delta = \operatorname{tr}\nabla^2$ and derive a Laplacian chain rule on functions:

$$\begin{aligned} \Delta(f\circ h) &= \operatorname{tr}\nabla((f'\circ h)\nabla h) = \operatorname{tr}((f'\circ h)\nabla^2 h + (f''\circ h)\langle\nabla h, \nabla h\rangle) \\ &= (f'\circ h)\Delta h + (f''\circ h)|\nabla h|^2. \end{aligned} \tag{4.12}$$

Taking $f = (\cdot)^{1/2}$ and $h = |A|^2$, this gives

$$\begin{aligned} L|A| &= \Delta|A| + \left(|A|^2 + \frac{1}{2}\right)|A| - \frac{1}{2}\langle x, \nabla|A|\rangle \\ &= \frac{\Delta|A|^2}{2|A|} - \frac{|\nabla|A|^2|^2}{4|A|^3} + \left(|A|^2 + \frac{1}{2}\right)|A| - \frac{1}{4}\left\langle x, \frac{\nabla|A|^2}{|A|}\right\rangle \\ &= \frac{2\langle A, \Delta A\rangle + 2|\nabla A|^2}{2|A|} - \frac{4|A|^2|\nabla|A||^2}{4|A|^3} + \left(|A|^2 + \frac{1}{2}\right)|A| - \frac{1}{2|A|}\langle A, \nabla_x A\rangle \\ &= \frac{\langle A, LA\rangle}{|A|} + \frac{|\nabla A|^2 - |\nabla|A||^2}{|A|}. \end{aligned}$$

Since $LA = A$, this gives the equality in (4.9). The inequality is due to the elementary *Kato inequality* $|\nabla|A|| \leq |\nabla A|$. □

The second ingredient we need is an estimate whose proof is a simple calculation and we omit.

Lemma 4.7. *For all hypersurfaces, it holds at points with $|A| \neq 0$ that*

$$\left(1+\frac{2}{n+1}\right)|\nabla|A||^2 \leq |\nabla A|^2 + \frac{2n}{n+1}|\nabla H|^2.$$

For the rest of this section, we reserve the notation $[\cdot]$ for weighted integrals over Σ, i.e.

$$[f] = \int_\Sigma f e^{-\frac{|x|^2}{4}}.$$

The next three lemmas are integral estimates for curvature-related quantities on shrinkers with positive mean curvature. The Peter-Paul inequality, $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ for any $\varepsilon > 0$, will be used many times in the proofs.

Lemma 4.8. *If $H > 0$ on Σ, then for any $\phi \in W^{1,2}$ and $\varepsilon > 0$ it holds that*

$$[(1-\varepsilon)\phi^2|\nabla \log H|^2 + \phi^2|A|^2] \leq \left[\frac{1}{\varepsilon}|\nabla\phi|^2 + \frac{1}{2}\phi^2\right].$$

Proof. Recalling that $\mathcal{L} = \Delta - \frac{1}{2}\langle x, \nabla(\cdot)\rangle$, apply (4.12) and $LH = H$ to get

$$\begin{aligned} \mathcal{L}\log H &= \frac{\Delta H}{H} - \frac{|\nabla H|^2}{H^2} - \frac{1}{2}\frac{\langle x, \nabla H\rangle}{H} = -|\nabla \log H|^2 + \frac{\Delta H - \frac{1}{2}\langle x, \nabla H\rangle}{H} \\ &= \frac{LH - |A|^2 H - \frac{1}{2}H}{H} - |\nabla \log H|^2 = \frac{1}{2} - |A|^2 - |\nabla \log H|^2. \end{aligned} \tag{4.13}$$

Thus, for any compactly supported smooth function η, we have $\eta \in W^{1,2}$ so Proposition 4.4 gives

$$[\langle \nabla\eta^2, \nabla \log H\rangle] = -[\eta^2 \mathcal{L}\log H] = \left[\eta^2(|A|^2 - \frac{1}{2} + |\nabla \log H|^2)\right]. \tag{4.14}$$

At the same time, we use Cauchy–Schwarz and Peter-Paul to get

$$|\langle \nabla\eta^2, \nabla \log H\rangle| = 2|\langle \nabla\eta, \eta\nabla \log H\rangle| \leq \frac{1}{\varepsilon}|\nabla\eta|^2 + \varepsilon\eta^2|\nabla \log H|^2 \tag{4.15}$$

for any $\varepsilon > 0$. Joining (4.14) and (4.15), we obtain

$$[(1-\varepsilon)\eta^2|\nabla \log H|^2 + \eta^2|A|^2] \leq \left[\frac{1}{\varepsilon}|\nabla\eta|^2 + \frac{1}{2}\eta^2\right]. \tag{4.16}$$

Let η_r be one on B_r and decay linearly to zero from ∂B_r to ∂B_{r+1}. Applying (4.16) to $\eta = \eta_r\phi$, taking $r \to \infty$ and applying the monotone convergence theorem gives the result. □

The following lemma adapts Schoen, Simon and Yau's integral curvature estimates for stable minimal surfaces [SSY75] to strictly mean convex shrinkers. The proof strategy is the same; see also [CM11, Theorem 2.21].

Lemma 4.9. *If $H > 0$ on Σ, then $[|A|^2 + |A|^4 + |\nabla|A||^2 + |\nabla A|^2] < \infty$.*

Proof. The hard part is proving $[|A|^4] < \infty$, so we will do this first. Let ϕ be a compactly supported smooth function on Σ. Applying Lemma 4.8 with $\varepsilon = 1$, we get a 'stability inequality'[2]

$$[\phi^2|A|^2] \leq \left[|\nabla\phi|^2 + \frac{1}{2}\phi^2\right].$$

Set $\phi = \eta|A|$ where η is smooth with compact support, $0 \leq \eta \leq 1$, and $|\nabla\eta| \leq 1$. Then

$$\begin{aligned} [\eta^2|A|^4] &\leq \left[|\nabla(\eta|A|)|^2 + \frac{1}{2}\eta^2|A|^2\right] \\ &\leq \left[\eta^2|\nabla|A||^2 + 2|\eta||\nabla|A|| \cdot |\nabla\eta||A| + |\nabla\eta|^2|A|^2 + \frac{1}{2}\eta^2|A|^2\right] \\ &\leq (1+\varepsilon)[\eta^2|\nabla|A||^2] + \left[|A|^2\left((1+\varepsilon^{-1})|\nabla\eta|^2 + \frac{1}{2}\eta^2\right)\right] \\ &\leq (1+\varepsilon)[\eta^2|\nabla|A||^2] + \left[\left(\frac{3}{2} + \frac{1}{\varepsilon}\right)|A|^2\right], \end{aligned} \tag{4.17}$$

where $\varepsilon > 0$ is arbitrary, and the third inequality uses Peter-Paul with $\eta|\nabla|A||$ and $|\nabla\eta||A|$. We will bound the first term on the right by a small multiple of $[\eta^2|A|^4]$ and other harmless terms; this will come from a Simons-type inequality which we now derive.

For a general function f, one has

$$\mathcal{L}f^2 = 2f\Delta f + 2|\nabla f|^2 - \frac{1}{2}\langle x, \nabla f^2\rangle = 2|\nabla f|^2 + 2f\mathcal{L}f.$$

Together with $\mathcal{L} = L - |A|^2 - \frac{1}{2}$, the equality in Lemma 4.6, and Lemma 4.7, we get

$$\begin{aligned} \mathcal{L}|A|^2 &= 2|\nabla|A||^2 + 2|A|\left(L|A| - |A|^3 - \frac{1}{2}|A|\right) \\ &= 2|\nabla A|^2 + |A|^2 - 2|A|^4 \\ &\geq 2\left(1 + \frac{2}{n+1}\right)|\nabla|A||^2 - \frac{4n}{n+1}|\nabla H|^2 + |A|^2 - 2|A|^4, \end{aligned} \tag{4.18}$$

which is our Simons-type inequality.[3] Integrating against $\frac{1}{2}\eta^2$, we have

$$\left[\frac{1}{2}\eta^2\mathcal{L}|A|^2\right] \geq \left[\left(1 + \frac{2}{n+1}\right)\eta^2|\nabla|A||^2 - \frac{2n}{n+1}|\nabla H|^2 - \eta^2|A|^4\right]. \tag{4.19}$$

Using Proposition 4.4 followed by Cauchy–Schwarz and Peter-Paul with $\eta|\nabla|A||$ and $|A||\nabla\eta|$, and finally the assumption $|\nabla\eta| \leq 1$, the left-hand side satisfies

$$\left[\frac{1}{2}\eta^2\mathcal{L}|A|^2\right] = -\left[\frac{1}{2}\langle\nabla\eta^2, \nabla|A|^2\rangle\right] = -[2\langle\eta\nabla\eta, |A|\nabla|A|\rangle] \leq [\varepsilon\eta^2|\nabla|A||^2 + \varepsilon^{-1}|A|^2], \tag{4.20}$$

where $\varepsilon > 0$ is again arbitrary. Substituting this into (4.19) and rearranging gives

$$[\eta^2|A|^4] + \left[\frac{2n}{n+1}|\nabla H|^2 + \frac{1}{\varepsilon}|A|^2\right] \geq \left(1 + \frac{2}{n+1} - \varepsilon\right)[\eta^2|\nabla|A||^2]. \tag{4.21}$$

[2] The stability inequality for minimal hypersurfaces is $\int_\Sigma \phi^2|A|^2 \leq \int_\Sigma |\nabla\phi|^2$, cf. [CM11, Lemma 1.32].
[3] Simons' inequality is $\Delta|A|^2 \geq 2\left(1 + \frac{2}{n}\right)|\nabla|A||^2 - 2|A|^4$, cf. [Sim68] or [CM11, Lemma 2.1].

Combining (4.17) and (4.21), we now get

$$[\eta^2|A|^4] \leq \frac{1+\varepsilon}{1+\frac{2}{n+1}-\varepsilon}\left\{[\eta^2|A|^4] + \left[\frac{2n}{n+1}|\nabla H|^2 + \frac{1}{\varepsilon}|A|^2\right]\right\} + \left[\left(\frac{3}{2}+\frac{1}{\varepsilon}\right)|A|^2\right]$$
$$\leq \frac{1+\varepsilon}{1+\frac{2}{n+1}-\varepsilon}[\eta^2|A|^4] + C[|\nabla H|^2 + |A|^2],$$

where $C = C(n, \varepsilon)$. Select $\varepsilon = \frac{1}{2(n+1)}$, so that $\frac{1+\varepsilon}{1+\frac{2}{n+1}-\varepsilon} < 1$. Then

$$[\eta^2|A|^4] \leq C[|\nabla H|^2 + |A|^2] \leq C[|A|^2(1+|x|^2)], \tag{4.22}$$

where $C = C(n)$ and the last inequality uses $|\nabla H| \leq |A||x|$ from (4.10). Next, the polynomial volume growth gives that $\sqrt{1+|x|^2} \in W^{1,2}$. To see this, we have $\nabla(\sqrt{1+|x|^2}) = \frac{x^\top}{\sqrt{1+|x|^2}}$ where $x^\top$ is the projection of x onto $T_x\Sigma$, so reasoning as in (4.1),

$$\left\|\sqrt{1+|x|^2}\right\|_{W^{1,2}} = \int_\Sigma \left(1+|x|^2+\frac{|x^\top|^2}{1+|x|^2}\right)e^{-\frac{|x|^2}{4}} \leq \int_\Sigma (2+|x|^2)e^{-\frac{|x|^2}{4}} < \infty.$$

Hence Lemma 4.8 applies with $\phi = \sqrt{1+|x|^2}$ and $\varepsilon = 1$ to give

$$[|A|^2(1+|x|^2)] \leq \left[\left|\nabla(\sqrt{1+|x|^2})\right|^2 + \frac{1}{2}(1+|x|^2)\right] \leq \left\|\sqrt{1+|x|^2}\right\|_{W^{1,2}} < \infty.$$

By (4.22), we therefore have $[\eta^2|A|^4] < \infty$. Taking a sequence of η's and using monotone convergence (as in the end of the proof of Lemma 4.8), we have $[|A|^4] < \infty$.

It follows that $[|A|^2] < \infty$. To establish $[|\nabla|A||^2] < \infty$, use (4.21) with $\varepsilon = 1$, the finiteness of $[|A|^4]$ and $[|A|^2]$, as well as $|\nabla H|^2 \leq |A|^2|x|^2$. Monotone convergence again gives the bound. It remains to show $[|\nabla A|^2] < \infty$. By integrating the second equality in (4.18) against $\frac{1}{2}\eta^2$, we get

$$\left[\frac{1}{2}\eta^2\mathcal{L}|A|^2\right] = \left[\eta^2|\nabla A|^2 + \frac{1}{2}\eta^2|A|^4 - \eta^2|A|^4\right] \geq \left[\eta^2(|\nabla A|^2 - |A|^4)\right].$$

Following (4.20) but stopping short of Peter-Paul, we have $[\frac{1}{2}\eta^2\mathcal{L}|A|^2] \leq 2[|A||\nabla|A||]$. Thus,

$$[\eta^2(|\nabla A|^2 - |A|^4)] \leq 2[|A||\nabla|A||] \leq [|A|^2 + |\nabla|A||^2].$$

Since we already have $[|A|^2 + |\nabla|A||^2 + |A|^4] < \infty$, the bound $[|\nabla A|^2] < \infty$ follows again from monotone convergence. □

The final lemma before the proof of Theorem 4.1 gives integral estimates which are needed to justify various applications of Proposition 4.4(ii) in the proof.

Lemma 4.10. *If $H > 0$ on Σ, then the following hold:*

$$[|A|^2|\nabla \log H| + |\nabla|A|^2| \cdot |\nabla \log H| + |A|^2|\mathcal{L}\log H|] < \infty, \tag{4.23}$$
$$[|A| \cdot |\nabla|A|| + |\nabla|A||^2 + |A| \cdot |\mathcal{L}|A||] < \infty. \tag{4.24}$$

Proof. Since $H > 0$, Lemma 4.9 gives

$$[|A|^2 + |A|^4 + |\nabla|A||^2 + |\nabla A|^2] < \infty. \tag{4.25}$$

Since $|\nabla|A|^2|^2 \leq 2|A|^2 + 2|\nabla|A||^2$ by the chain rule and Peter-Paul, the first and third terms of (4.25) show that $|A| \in W^{1,2}$. We may then apply Lemma 4.8 with $\phi = |A|$, $\varepsilon = 1/2$ to get

$$[|A|^2|\nabla \log H|^2] \leq 2\left[2|\nabla|A||^2 + \frac{1}{2}|A|^2\right] \leq 4\left[|\nabla|A||^2 + |A|^2\right] < \infty, \tag{4.26}$$

the last inequality being (4.25). Peter-Paul then yields

$$[|\nabla|A|^2| \cdot |\nabla \log H|] = 2[|A| \cdot |\nabla|A|| \cdot |\nabla \log H|] \leq [|A|^2|\nabla \log H|^2 + |\nabla|A||^2] < \infty. \tag{4.27}$$

Also, (4.13) and (4.26) together give us that

$$[|A|^2|\mathcal{L} \log H|] \leq \left[\frac{1}{2}|A|^2 + |A|^4 + |A|^2|\nabla \log H|^2\right] < \infty. \tag{4.28}$$

Adding (4.26), (4.27) and (4.28) gives (4.23). Next, using (4.8) and the equality in Lemma 4.6,

$$|A|\mathcal{L}|A| = |A|\left(L|A| - |A|^2|A| - \frac{1}{2}|A|\right) = \frac{1}{2}|A|^2 - |A|^4 + |\nabla A|^2 - |\nabla|A||^2.$$

Using this, Peter-Paul and (4.25), we arrive at (4.24):

$$[|A| \cdot |\nabla|A|| + |\nabla|A||^2 + |A| \cdot |\mathcal{L}|A||] \leq \left[\frac{1}{2}|A|^2 - \frac{1}{2}|\nabla|A||^2 + \frac{1}{2}|A|^2 - |A|^4 + |\nabla A|^2\right] < \infty.$$

□

Proof of Theorem 4.1. It suffices to prove the theorem assuming Σ is connected. If not, then since every connected component of Σ will be one of the hypersurfaces listed in the theorem, and any two such hypersurfaces intersect, this contradicts the fact that Σ is smooth and embedded.

Starting from (4.11), use Codazzi's equations $\nabla_i h_{jl} = \nabla_l h_{ij}$ then trace both sides with g^{ij} to get

$$\Delta H = \frac{1}{2}\langle x, e_l\rangle \nabla_l H + \frac{1}{2}H - |A|^2 H \leq H + \frac{1}{2}\langle x, \nabla H\rangle,$$

where the inequality uses $H \geq 0$. By Theorem 2.5 and the fact that $H \geq 0$, we have that either $H = 0$ everywhere or $H > 0$ everywhere. If $H = 0$ everywhere, then the shrinker equation is $\langle x, \mathbf{n}\rangle = 2H = 0$, so x is a tangent vector field on Σ. Pick any $p \in \Sigma$ and solve for the flow of x starting from p. The flow line traces the ray $\{\lambda p \mid \lambda > 0\}$, so Σ is a cone.[4] Being smooth, Σ must be a hyperplane through the origin, i.e. a rotation of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$.

We may therefore assume $H > 0$, proceeding in two steps. Firstly, we use Lemma 4.10 to bring us to the key identity $\nabla_k h_{ij} = c_k h_{ij}$, where c_k depends only on k and the point on Σ. Because the indices on the left can be permuted by Codazzi's equations, this indicates that Σ has a high degree of symmetry, consistent with the theorem. The second step follows Huisken [Hui93], where this identity is massaged into a full classification of the possibilities for Σ.

[4]The rays do not include the origin, but Σ is a closed subset of $\mathbb{R}^{n+1}$ so it must contain the origin.

Step 1: The key geometric identity. The first statement in Lemma 4.10 allows us to apply Proposition 4.4 to $|A|^2$ and $\log H$. Doing this then using (4.13), we get

$$[\langle \nabla |A|^2, \nabla \log H\rangle] = -[|A|^2 \mathcal{L} \log H] = \left[|A|^2 \left(|A|^2 - \frac{1}{2} + |\nabla \log H|^2 \right) \right].$$

Similarly, the first statement in Lemma 4.10 allows us to apply Proposition 4.4 to two copies of $|A|$. Doing this then using $L|A| \geq |A|$ from Lemma 4.6, we get

$$[|\nabla |A||^2] = -[|A| \mathcal{L} |A|] = -\left[|A| \left(L|A| - |A|^2|A| - \frac{1}{2}|A| \right) \right] \leq \left[|A|^4 - \frac{1}{2}|A|^2 \right].$$

Combining these two gives

$$[\langle \nabla |A|^2, \nabla \log H\rangle] \geq [|A|^2 |\nabla \log H|^2 + |\nabla |A||^2],$$

or

$$0 \geq [|A|^2|\nabla \log H|^2 - 2\langle \nabla |A|, |A|\nabla \log H\rangle + |\nabla |A||^2] = \left[\Big| |A|\nabla \log H - \nabla |A| \Big|^2 \right].$$

Therefore, $|A|\nabla \log H - \nabla |A|$ vanishes identically on Σ. Rewriting this as $\nabla \log H = \nabla \log |A|$ and integrating out, we get $H = \beta |A|$, where $\beta : \Sigma \to \mathbb{R}$ is positive. In particular $L|A| = \beta L H = \beta H = |A|$, so by Lemma 4.6 we have $|\nabla A|^2 = |\nabla |A||^2$. This implies $\nabla_k h_{ij} = c_k h_{ij}$ for each $i, j, k = 1, \ldots, n$. Contracting with g^{ij}, we also immediately get $c_k = \frac{\nabla_k H}{H}$.

Step 2: Squeezing out a classification. Pick a point $p \in \Sigma$, and work in a coordinate chart where at p we have $h_{ij} = \lambda_i \delta_{ij}$ for some constants λ_i. Then $\nabla_k h_{ij} = c_k h_{ij} = c_k \lambda_i \delta_{ij}$, which vanishes if $i \neq j$. By Codazzi's equations, we even have that $\nabla_k h_{ij} = 0$ unless $i = j = k$. Note that if $\lambda_i \neq 0$ and $j \neq i$, then $0 = \nabla_j h_{ii} = c_j \lambda_i$, so $c_j = 0$. Thus, if two or more of the λ_i's are zero, then all of the c_j's are zero, hence $\nabla A = 0$ at p. We have thus shown that

($*$) If $A(p)$ has rank two or greater, then $\nabla A(p) = 0$.

We therefore divide into two cases depending on the rank of A at p.

Case 1: The rank is at least two. We will show that $\mathrm{rank}(A) \geq 2$ *everywhere*. Where $q \in \Sigma$ is arbitrary, let $\lambda_1(q)$ and $\lambda_2(q)$ be the two eigenvalues of $A(q)$ of largest absolute value, and define

$$\Sigma_2 = \{q \in \Sigma \mid \lambda_1(q) = \lambda_1(p) \text{ and } \lambda_2(q) = \lambda_2(p)\}.$$

Since λ_1 and λ_2 are continuous and Σ_2 is defined by a closed condition, Σ_2 is closed. Since $\lambda_1(p), \lambda_2(p) \neq 0$, for any point $q \in \Sigma_2$ we have $\mathrm{rank}(A(q)) \geq 2$. This is an open condition, so there exists an open set Ω containing q where $\mathrm{rank}(A(\cdot)) \geq 2$. But this implies $\nabla A = 0$ on Ω, so the eigenvalues of A are constant on Ω. It follows that $\Omega \subset \Sigma_2$, so Σ_2 is open. Now Σ_2 is a closed and open subset of the connected set Σ, hence $\Sigma_2 = \Sigma$.

It follows by ($*$) that $\nabla A = 0$ over all of Σ. Using Theorem 4 of Lawson [Law69], Σ splits isometrically as some rotation of $S^k_r \times \mathbb{R}^{n-k}$. For such a cylinder, $H = \frac{k}{r}$. On the other hand, the shrinker equation gives $H = \frac{\langle x, \mathbf{n}\rangle}{2} = \frac{r}{2}$. Equating these yields $r = \sqrt{2k}$.

Case 2: The rank is one. From above, the rank of A must be one everywhere on Σ, so the only nonzero eigenvalue is $-H$, as $H = -\operatorname{tr} A$. For each $p \in \Sigma$, let $V(p) \in T_p\Sigma$ be a unit $(-H)$-eigenvector for A. As A is smooth, V is a smooth vector field at least locally (globally it is defined up to a sign). Regardless of sign, at each tangent space we have

$$A(v,w) = A(\langle v,V\rangle V, \langle w,V\rangle V) = -H\langle v,V\rangle \langle w,V\rangle . \tag{4.29}$$

Fix some $p \in \Sigma$, choose a frame $\{e_i\}_{i=1}^n$ such that $e_1(p) = V(p)$ and the matrix of $A(p)$ in this frame is $\mathrm{diag}(-H(p), 0, \ldots, 0)$. Then $\nabla_k h_{ij} = 0$ except possibly when $i = j = k = 1$, so at p,

$$\nabla_u A(v,w) = \nabla_{\langle u,V\rangle V} A(\langle v,V\rangle V, \langle w,V\rangle V) = -\frac{\nabla_V A(V,V)}{H}\langle u,V\rangle A(v,w), \tag{4.30}$$

where the second equality uses linearity and (4.29). Since p was arbitrary and the right-hand side of (4.30) is unchanged under substituting $V \mapsto -V$, this identity holds at every point. Taking a unit speed geodesic $\gamma(t)$ on Σ and a parallel vector field $v(t)$ on Σ along $\gamma(t)$, this gives

$$\frac{d}{dt}A(v(t),v(t)) = \nabla_{\dot\gamma(t)} A(v(t),v(t)) = -\frac{\nabla_V A(V,V)}{H}\langle v(t),V\rangle A(v(t),v(t)), \tag{4.31}$$

where H and V are evaluated at $\gamma(t)$. Selecting $v(0)$ to be in the kernel of A at $\gamma(0)$, then applying Lemma 2.7 to (4.31), we find that $A(v,v)$ vanishes along $\gamma(t)$. Using (4.30), we have that at $\gamma(t)$,

$$\left\langle \overline{\nabla}_{\dot\gamma} v, \mathbf{n}\right\rangle = \langle \nabla_{\dot\gamma} v + A(\dot\gamma, v)\mathbf{n}, \mathbf{n}\rangle = A(\dot\gamma, v) = 0. \tag{4.32}$$

At the same time, v being parallel along γ implies that for any tangent vector X,

$$\left\langle \overline{\nabla}_{\dot\gamma} v, X\right\rangle = \langle \nabla_{\dot\gamma} v, X\rangle = \langle 0, X\rangle = 0. \tag{4.33}$$

From (4.32) and (4.33), we see that $v(t)$ is actually constant along γ. By the arbitrariness of $\gamma(t)$ and $v(t)$ and recalling that A has an $(n-1)$-dimensional kernel everywhere, we conclude that there are $n-1$ constant vectors $e_2, \ldots, e_n$ tangent to Σ constituting a global orthonormal frame for the kernel of A. These directions are precisely the noncurved ones, so translations in these directions are isometries of Σ. Thus, Σ splits isometrically as a product $K \times \widetilde{\gamma}$, where $K = \mathrm{span}\{e_2, \ldots, e_n\}$ and $\widetilde{\gamma} \subset \mathbb{R}^2$ is a smooth, embedded, convex shrinker in $\mathbb{R}^2$ with polynomial length growth. A short argument ([CM12, Lemma 10.39]) followed by the Gage-Hamilton theorem [GH86] shows that $\widetilde{\gamma}$ must be a round circle.[5] Arguing as in the end of Case 1, the radius must be $\sqrt{2}$. □

[5]See [Man11, Proposition 3.4.1] for another way of proving this part.

Chapter 5

Uniqueness of Compact Tangent Flows

From the discussion in §3.4, the utility of a shrinker classification result for singularity analysis in MCF lays contingent on the uniqueness of tangent flows. In this chapter, we will prove Schulze's result [Sch14] that uniqueness indeed holds for compact, embedded tangent flows.

Theorem 5.1 ([Sch14])**.** *Let Σ be a unit multiplicity tangent flow arising from a compact, embedded MCF. If Σ is smooth, compact and embedded, then it is the unique tangent flow at that point.*

When the MCF is also mean convex, Theorem 5.1 implies uniqueness of spherical tangent flows, which are the only compact blowups by Corollary 4.2. The theorem however has no convexity assumptions, so it holds for peculiar examples like Angenent's shrinking torus [Ang92] whenever it arises as a tangent flow. In fact, Schulze proved Theorem 5.1 for MCF and Brakke flows in arbitrary codimension. The proof is easily adapted from the unit codimension case, but we will exclude this as we have not formally introduced these flows.

In §5.1, we discuss Łojasiewicz inequalities and their relation to uniqueness problems. To tackle uniqueness of tangent flows, we need to generalise these to infinite-dimensional 'spaces of hypersurfaces'. This is done in §5.2, where we follow Simon's influential paper [Sim83a] to prove the now-called Łojasiewicz–Simon gradient inequality. In §5.3, we will see how Schulze used this inequality to prove Theorem 5.1.

5.1 Łojasiewicz inequalities and uniqueness

Between the late 1950s and early 1960s, Łojasiewicz proved a collection of deep results in real algebraic geometry concerning real-analytic functions and their gradient flows [Łoj63, Łoj65]. Let $f : U \to \mathbb{R}$ be real-analytic on an open subset U of $\mathbb{R}^n$, and denote by Z the zero set of f, assumed nonempty. The first of these results states that for any compact subset K of U, there exist $\alpha \geq 2$ and a positive constant C_1 such that for all $x \in K$,

$$\inf_{z \in Z} |x - z|^\alpha \leq C_1 |f(x)|. \tag{5.1}$$

Using this inequality, Łojasiewicz further proved that if $p \in U$ is a critical point of f, then there exists a neighbourhood W of p and constants $\beta \in (\frac{1}{2}, 1)$ and $C_2 > 0$ such that for all $x \in W$,

$$|f(x) - f(p)|^\beta \leq C_2 |\nabla f(x)|. \tag{5.2}$$

Łojasiewicz used this second inequality (the *gradient inequality*) to prove the *Łojasiewicz theorem*: with the same assumptions on f, suppose $\gamma : [0, \infty) \to \mathbb{R}^n$ is a negative gradient flow line of f. That is, $\gamma'(t) = -\nabla f(\gamma(t))$. If γ has an accumulation point x_∞, then γ has finite length and

$$\lim_{t \to \infty} \gamma(t) = x_\infty.$$

In other words, x_∞ is the unique limit of γ as $t \to \infty$.

How Łojasiewicz's results relate to uniqueness of tangent flows draws from an important observation: the RMCF $\widetilde{\varphi}$ is the negative gradient flow of the $\mathcal{F}$-functional. To make this precise, let Σ_s be the image of $\widetilde{\varphi}(\cdot, s)$, and let $\mathcal{M}^\nu_s$ be the $L^2(\nu)$ Euler–Lagrange functional of $\mathcal{F}_{\Sigma_s}$. That is,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}_{\Sigma_s}(u + tv) = -(4\pi)^{-\frac{n}{2}} \int_{\Sigma_s} v \mathcal{M}^\nu_s(u) e^{-\frac{|x|^2}{4}}.$$

From (4.3), we see that $\mathcal{M}^\nu_s(0) = -H + \frac{\langle x, \mathbf{n} \rangle}{2}$. But this is equal to $\frac{\partial \widetilde{\varphi}}{\partial s}$ by Lemma 3.21. Thus, at all times, $\frac{\partial \widetilde{\varphi}}{\partial s}$ is the negative $L^2(\nu)$-gradient of $\mathcal{F}_{\Sigma_s}$ at the zero function. Viewing $\mathcal{F}$ as the Gaussian area functional on the 'space of hypersurfaces', this means $\widetilde{\varphi}$ flows by the negative gradient of $\mathcal{F}$.

The idea is to work in a space whose points are hypersurfaces in $\mathbb{R}^{n+1}$, so that $\mathcal{F}$ is defined on this space. The RMCF is the negative gradient flow of $\mathcal{F}$, and traces a curve in this space whose accumulation points are its tangent flows. If the Łojasiewicz theorem generalises to functionals on this space, then applying it to $\mathcal{F}$ results in uniqueness of tangent flows. See Figure 5.1.

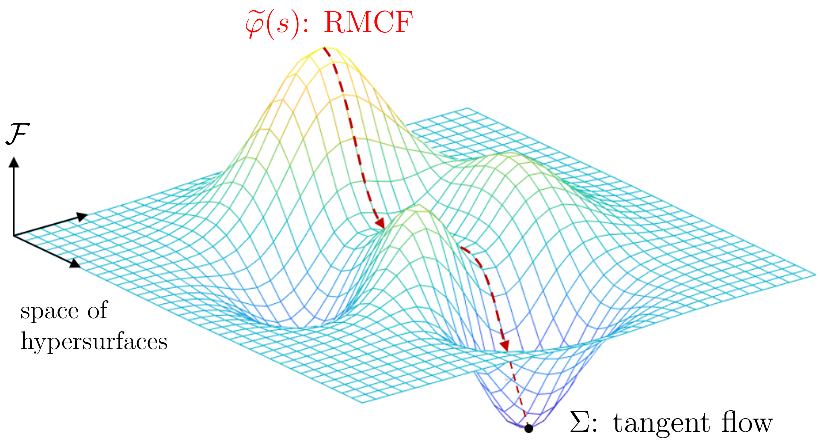

Figure 5.1: RMCF is the negative gradient flow of $\mathcal{F}$ in the space of hypersurfaces. We want to generalise the Łojasiewicz theorem to functionals on this space (in particular $\mathcal{F}$) to prove that tangent flows, like Σ here, are unique.

Any reasonable manifestation of the 'space of hypersurfaces' must be infinite-dimensional, so we are led to ask which infinite-dimensional spaces admit generalisations of Łojasiewicz's results. Leon Simon [Sim83a] made the first stride in this direction, generalising the Łojasiewicz inequalities to Hölder sections of vector bundles over a compact Riemannian manifold M. In the same paper, he used these *Łojasiewicz–Simon inequalities* (as they are now named) to generalise the Łojasiewicz theorem and exhibited a number of geometric applications, most notably to prove a long-standing conjecture about uniqueness of tangent cones for minimal hypersurfaces. Thirty years later, Schulze recognised the opportunity to apply Simon's results in the setting of mean curvature flow, giving Theorem 5.1.

In view of our applications, we will prove the Łojasiewicz–Simon gradient inequality for real-valued functions on M. For the proof, we will take the original gradient inequality (5.2) as given. This is because the original Łojasiewicz inequalities are known to be notoriously difficult to prove, requiring specialised machinery well beyond the scope of this thesis. Unlike the previous chapter, all L^p and Sobolev norms in this chapter are unweighted.[1]

5.2 The Łojasiewicz–Simon gradient inequality

In this section, (M, g) is a compact n-dimensional Riemannian manifold with Levi-Civita connection ∇. Let $\mathcal{E} : C^1(M) \to \mathbb{R}$ be a functional satisfying Assumption 5.2 below.

Assumption 5.2. *The functional $\mathcal{E} : C^1 \to \mathbb{R}$ can be written as*

$$\mathcal{E}(u) = \int_M E(p, u(p), \nabla u(p))$$

for some smooth real-valued function E of (p, q, z) where $p \in M$, $q \in \mathbb{R}$ and $z \in T_pM$. Also,

(i) For each $p \in M$, we have that E is uniformly convex in the z variable when $q = 0$. That is,

$$\frac{d^2}{ds^2}\Big|_{s=0} E(p, 0, sz) \geq c|z|^2, \qquad \forall z \in T_pM. \tag{5.3}$$

with $c > 0$ independent of p, z.

(ii) There exists $\delta > 0$ such that for each p, the dependence of $E(p, q, z)$ on (q, z) is real-analytic whenever $|q|, |z| < \delta$.

Let $\mathcal{M}$ be the (unweighted) Euler–Lagrange functional of $\mathcal{E}$, defined in (2.11). From §A.1, uniform convexity (5.3) implies that the linearisation L of $\mathcal{M}$ at 0 is a symmetric, uniformly elliptic second-order operator with smooth coefficients. By 2.2, the kernel $\mathcal{K}$ of L is finite-dimensional and has an L^2-orthonormal basis of smooth eigenfunctions $\varphi_1, \ldots, \varphi_d$. We will let Π and $(\cdot)^\perp$ denote L^2-orthogonal projection onto $\mathcal{K}$ and $\mathcal{K}^\perp$ respectively.

Analyticity of E allows us to make sense of $E(p, q, z)$ for $q \in \mathbb{C}$ and $z \in T_pM_\mathbb{C}$ where $|q|, |z| < \delta$, and the subscript $\mathbb{C}$ denotes complexification. Moreover, the dependence on (q, z) is holomorphic in this region. Using the explicit expression (A.1) for $\mathcal{M}$, we extend $\mathcal{M}$ to complex-valued functions $u \in C^{2,\alpha}_\mathbb{C}$ with $\|u\|_{C^1_\mathbb{C}} < \delta$. In particular, $\mathcal{M}$ is defined on $C^{2,\alpha}_\mathbb{C} \cap B_\delta(0)$.

Under these assumptions, Simon's generalisation of (5.1) and (5.2) becomes possible. We will only prove the analogue of the latter, which is what we need later. We fix an arbitrary constant $\alpha \in (0, 1)$ for the rest of this chapter.

[1]This may seem strange given that we motivated the use of Łojasiewicz inequalities using the weighted spaces, but this is more of an artefact of the analysis than a conceptual discrepancy.

Theorem 5.3 ([Sim83a])**.** *Suppose $\mathcal{M}(0) = 0$. There is a neighbourhood U of the origin in $C^{2,\alpha}$ and constants $\beta \in (\frac{1}{2}, 1)$ and $C > 0$ depending on n, M and the form of $\mathcal{E}$ such that if $u \in U$, then*

$$|\mathcal{E}(u) - \mathcal{E}(0)|^\beta \leq C \|\mathcal{M}(u)\|_{L^2}.$$

The proof we present amalgamates the ones in [Sim83a] and [Sim96]. Most of the work goes into the next lemma, which is part of a procedure called *Lyapunov-Schmidt reduction*.

Lemma 5.4. *There exists a neighbourhood U of the origin in $C^{2,\alpha}$, a neighbourhood W of the origin in $C^{0,\alpha}$, and a real-analytic bijection $\Psi : W \to U$ such that $\Pi u \in W \cap K$ whenever $u \in U$, and*

$$\|\Psi f - \Psi g\|_{W^{2,2}} \leq C \|f - g\|_{L^2}, \qquad \forall f, g \in W, \tag{5.4}$$

$$\|\Psi \Pi u - \Pi u\|_{W^{2,2}} \leq C \|\Pi u\|_{L^2}^2, \qquad \forall u \in U. \tag{5.5}$$

Here C depends on n, M and the form of $\mathcal{E}$. Moreover, if we define $\widetilde{W} = \{\xi = (\xi^1, \dots, \xi^d) \in \mathbb{R}^d \mid \xi^j \varphi_j \in W \cap K\}$ and $f(\xi) = \mathcal{E}(\Psi(\xi^j \varphi_j))$, then

$$|\nabla f(\xi)| \leq 2 \left\|\mathcal{M}\Psi(\xi^j \varphi_j)\right\|_{L^2}, \qquad \forall \xi \in \widetilde{W}. \tag{5.6}$$

Remark 5.5. With a little more work (see [Sim83a]), we can show that

$$\{u \in U \mid \mathcal{M}(u) = 0\} = \Psi\left(\left\{\xi^j \varphi_j \mid \xi \in \widetilde{W} \text{ and } \nabla f(\xi) = 0\right\}\right), \tag{5.7}$$

so $Q = \Psi(W \cap K)$ contains all zeros of $\mathcal{M}$ near the origin. Since $W \cap K$ is finite-dimensional and consists of zeros of L (the derivative of $\mathcal{M}$) near the origin, Ψ is an 'exponential map' identifying zeros of L with zeros of $\mathcal{M}$ near the origin. This way, Q is a finite-dimensional submanifold of U (see Figure 5.2). This is the essence of Lyapunov-Schmidt reduction; it finitely parametrises the set of critical points of $\mathcal{E}$ (i.e. zeros of $\mathcal{M}$) near the origin. We would need to explicitly use (5.7) to generalise the first Łojasiewicz inequality (5.1), but this is not needed for Theorem 5.3.

To prove Theorem 5.3, we will decompose $|\mathcal{E}(u) - \mathcal{E}(0)|$ into an infinite-dimensional part and a finite-dimensional part by writing

$$|\mathcal{E}(u) - \mathcal{E}(0)| \leq |\mathcal{E}(u) - \mathcal{E}(\Psi\Pi u)| + |\mathcal{E}(\Psi\Pi u) - \mathcal{E}(0)|.$$

The original gradient inequality (5.2) controls the finite-dimensional part $|\mathcal{E}(\Psi\Pi u) - \mathcal{E}(0)|$, while a separate argument will show that the infinite-dimensional part is negligible. Note that (5.5) says $\Psi\Pi u \approx \Pi u$, so we have $Q \approx W \cap K$. Thus, the finite-dimensional part is almost $|\mathcal{E}(\Pi u) - \mathcal{E}(0)|$, a difference taken in the direction of $\mathcal{K}$, while the infinite-dimensional part is almost $|\mathcal{E}(u) - \mathcal{E}(\Pi u)|$, a difference taken in the direction of $\mathcal{K}^\perp$. This idea will be revisited in §6.4.

Proof of Theorem 5.3 assuming Lemma 5.4. Let U, W, $\widetilde{W}$, Ψ and f be as in Lemma 5.4. We may further assume that U is convex and $\|u\|_{C^{2,\alpha}} \leq 1$ for all $u \in U$. We first estimate the infinite-dimensional part $|\mathcal{E}(u) - \mathcal{E}(\Psi\Pi u)|$. Taking $u \in U$, we have $\Pi u \in W$. Then $\Psi\Pi u \in U$ and so

$$\begin{aligned} |\mathcal{E}(u) - \mathcal{E}(\Psi\Pi u)| &= \left|\int_0^1 \frac{d}{ds}\mathcal{E}(u + s(\Psi\Pi u - u))\, ds\right| \\ &= \left|\int_0^1 \langle \mathcal{M}(u + s(\Psi\Pi u - u)), \Psi\Pi u - u\rangle_{L^2}\, ds\right|, \end{aligned} \tag{5.8}$$

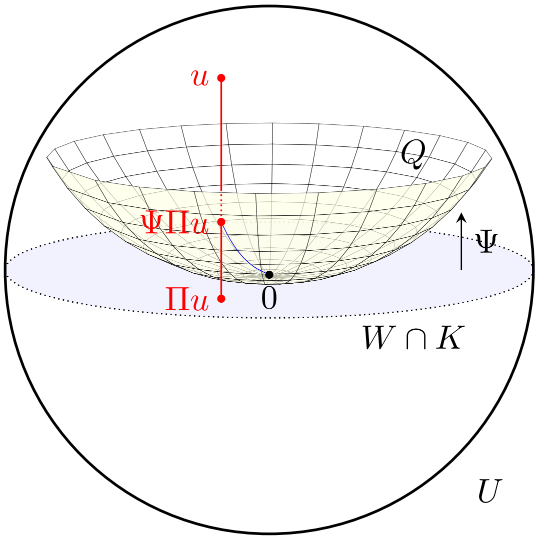

Figure 5.2: Ψ exponentiates $W \cap K$ as a finite-dimensional submanifold Q of U. We use this to split $\mathcal{E}(u) - \mathcal{E}(0)$ into finite-dimensional and infinite-dimensional parts.

by the definition of $\mathcal{M}$. Since $\|u + s(\Psi\Pi u - u)\|_{C^{2,\alpha}} \leq 1$ for all $s \in [0,1]$, Proposition A.3 gives

$$\|\mathcal{M}(u + s(\Psi\Pi u - u)) - \mathcal{M}(u)\|_{L^2} \leq Cs\,\|\Psi\Pi u - u\|_{W^{2,2}}\,,$$

where C is a constant depending on n, M and the form of $\mathcal{E}$. This dependence will remain, but C can change from line to line. Using the above in (5.8), we bound

$$\begin{aligned}|\mathcal{E}(u) - \mathcal{E}(\Psi\Pi u)| &\leq \int_0^1 \|\mathcal{M}(u + s(\Psi\Pi u - u)) - \mathcal{M}(u)\|_{L^2}\,\|\Psi\Pi u - u\|_{L^2}\,ds \\ &\quad + \int_0^1 \|\mathcal{M}(u)\|_{L^2}\,\|\Psi\Pi u - u\|_{L^2}\,ds \\ &\leq C\,\|\Psi\Pi u - u\|_{W^{2,2}}^2 + \|\mathcal{M}(u)\|_{L^2}\,\|\Psi\Pi u - u\|_{W^{2,2}} \\ &\leq C\,\|\mathcal{M}u\|_{L^2}^2\,,\end{aligned} \tag{5.9}$$

where the last inequality is (5.4) with $f = \Pi u, g = \mathcal{N}u$.

Now we estimate the finite-dimensional part $|\mathcal{E}(\Psi\Pi u) - \mathcal{E}(0)|$. We have $\Pi u = \xi^j \varphi_j$ for some $\xi \in \widetilde{W}$, so by (5.6), Proposition A.3 and (5.4),

$$\begin{aligned}|\nabla f(\xi)| &\leq 2\,\|\mathcal{M}(\Psi\Pi u)\|_{L^2} \leq 2\,\|\mathcal{M}u\|_{L^2} + 2\,\|\mathcal{M}(\Psi\Pi u) - \mathcal{M}u\|_{L^2} \\ &\leq 2\,\|\mathcal{M}u\|_{L^2} + C\,\|\Psi\Pi u - u\|_{W^{2,2}} \leq C\,\|\mathcal{M}u\|_{L^2}\,.\end{aligned}$$

Since f is analytic, (5.2) gives $\beta \in (\frac{1}{2}, 1)$ and C such that $|f(\xi) - f(0)|^\beta \leq C|\nabla f(\xi)|$. Thus

$$|\mathcal{E}(\Psi\Pi u) - \mathcal{E}(0)|^\beta = |f(\xi) - f(0)|^\beta \leq C|\nabla f(\xi)| \leq C\,\|\mathcal{M}u\|_{L^2}\,. \tag{5.10}$$

Since $\mathcal{M} : C^{2,\alpha} \to C^{0,\alpha}$ is continuous, $\mathcal{M}(0) = 0$, and U is contained in the unit ball in $C^{2,\alpha}$, there exists $\sigma > 0$ so that $\|\mathcal{M}u\|_{C^{0,\alpha}} \leq \sigma$. Since M is compact and has finite measure, we have

$$\|\mathcal{M}u\|_{L^2} \leq C\,\|\mathcal{M}u\|_{L^\infty} \leq C\,\|\mathcal{M}u\|_{C^{0,\alpha}} \leq C\sigma \leq C. \tag{5.11}$$

Using the elementary inequality $|a+b|^\beta \le 2^\beta(|a|^\beta + |b|^\beta)$, then (5.9), (5.10), (5.11), we get

$$\begin{aligned}|\mathcal{E}(u) - \mathcal{E}(0)|^\beta &\le C(|\mathcal{E}(u) - \mathcal{E}(\Psi\Pi u)|^\beta + |\mathcal{E}(\Psi\Pi u) - \mathcal{E}(0)|^\beta)\\ &\le C(\|\mathcal{M}u\|_{L^2}^{2\beta} + \|\mathcal{M}u\|_{L^2})\\ &\le C\,\|\mathcal{M}u\|_{L^2}.\end{aligned}$$

This is the required inequality. □

It remains to prove Lemma 5.4.

Proof of Lemma 5.4. We proceed in four parts. The first part constructs Ψ, and the next three parts prove (5.4), (5.5) and (5.6) in that order.

Step 1: The function Ψ. Define $\mathcal{N} : C^{2,\alpha} \to C^{0,\alpha}$ by

$$\mathcal{N}u = \Pi u + \mathcal{M}u. \tag{5.12}$$

Since Π is linear, the linearisation of $\mathcal{N}$ at 0 is

$$d\mathcal{N}|_0(v) = \Pi v + Lv,$$

which is symmetric and elliptic, just as L is. Note that $d\mathcal{N}|_0$ has trivial kernel on C^2; indeed if $d\mathcal{N}|_0(v) = 0$, then $\Pi v = -Lv$, but $\Pi v \in K$ while $-Lv \in K^\perp$, so we have $Lv = 0 = \Pi v$. But the first equality implies $v \in K$ while the second implies $v \in K^\perp$. Hence, $v = 0$. By Theorem 2.2, this implies $d\mathcal{N}|_0$ has trivial kernel on $W^{2,2}$, not just C^2. Thus, elliptic theory (Theorem 2.3 and Theorem 2.4) gives that $d\mathcal{N}|_0$ is an isomorphism from $C^{2,\alpha}$ to $C^{0,\alpha}$.

Viewing $\mathcal{M}$ as defined on $C^{2,\alpha}_{\mathbb{C}} \cap B_\delta(0)$, its linearisation exists at any point in this domain by the explicit formula for $\mathcal{M}$ in (A.1) and the holomorphicity of E noted earlier. Thus, $\mathcal{M}$ is holomorphic on $C^{2,\alpha}_{\mathbb{C}} \cap B_\delta(0)$, and so too is $\mathcal{N}$. Since $d\mathcal{N}|_0 : C^{2,\alpha} \to C^{0,\alpha}$ is an isomorphism from the last paragraph, $d\mathcal{N}|_0 : C^{2,\alpha}_{\mathbb{C}} \to C^{0,\alpha}_{\mathbb{C}}$ is also an isomorphism.

Applying an appropriate version of the inverse function theorem [Nir01, Theorem 2.7.2] to the holomorphic map $\mathcal{N} : C^{2,\alpha}_{\mathbb{C}} \cap B_\delta(0) \to C^{0,\alpha}_{\mathbb{C}}$, we get that $\mathcal{N}$ that bijects from a neighbourhood $U_{\mathbb{C}}$ of 0 in $C^{2,\alpha}_{\mathbb{C}} \cap B_\delta(0)$ onto a neighbourhood $W_{\mathbb{C}}$ of 0 in $C^{0,\alpha}_{\mathbb{C}}$, with holomorphic inverse $\Psi_{\mathbb{C}} = \mathcal{N}^{-1}$. Setting $U = C^{2,\alpha} \cap U_{\mathbb{C}}$ and $W = C^{2,\alpha} \cap W_{\mathbb{C}}$, the function $\Psi_{\mathbb{C}}$ restricts to a bijective real-analytic map $\Psi : W \to U$. Note that $U \subset C^{2,\alpha} \cap B_\delta(0)$.

In the next three steps, we successively shrink U and W so that (5.4)-(5.6) hold. Whenever we shrink U, it is understood that W is shrunk accordingly to keep $\Psi : W \to U$ a bijection, and vice versa when we shrink W. All constants C below depend on n, M and the form of $\mathcal{E}$.

Step 2: Establishing (5.4). Let $f, g \in W$ and write $u = \Psi f, v = \Psi g$. We have $u, v \in U$, so $\|u\|_{C^2}, \|v\|_{C^2} < \delta$. Thus, Proposition A.3 gives

$$\mathcal{M}u - \mathcal{M}v = L(u-v) + a^{ij}(u-v)_{ij} + b^\alpha(u-v)_\alpha + c(u-v),$$

where subscripts denote partial derivatives, and $|a| + |b| + |c| \le C(\|u\|_{C^2} + \|v\|_{C^2})$. Since $f = \mathcal{N}\Psi f = \Pi u + \mathcal{M}u$, and similarly $g = \Pi v + \mathcal{M}v$, the above identity can be written

$$\Pi(v-u) + L(v-u) = a^{ij}(u-v)_{ij} + b^\alpha(u-v)_\alpha + c(u-v) + g - f. \tag{5.13}$$

Projecting both sides onto $\mathcal{K}$ using the L^2 inner product, and noting that L takes values in $\mathcal{K}^\perp$ (Theorem 2.3), we get

$$\Pi(v-u) = \Pi F,$$

where F is the right-hand side of (5.13). Thus we obtain the L^2 estimate

$$\|\Pi(v-u)\|_{L^2} = \|\Pi F\|_{L^2} \le \|F\|_{L^2} \le C(\|u\|_{C^2} + \|v\|_{C^2})\,\|u-v\|_{W^{2,2}} + \|g-f\|_{L^2}\,.$$

But the finite-dimensionality of $\mathcal{K}$ means the L^2 and $W^{2,2}$ norms on $\mathcal{K}$ are equivalent, so we get

$$\|\Pi(v-u)\|_{W^{2,2}} \le C(\|u\|_{C^2} + \|v\|_{C^2})\,\|u-v\|_{W^{2,2}} + C\,\|g-f\|_{L^2}\,. \tag{5.14}$$

Similarly, projecting both sides of (5.13) onto $K^\perp$ gives

$$L((v-u)^\perp) = F^\perp.$$

Therefore, Theorem 2.4 comes in to give the estimate

$$\|(v-u)^\perp\|_{W^{2,2}} \le C\|F^\perp\|_{L^2} \le C\,\|F\|_{L^2} \le C(\|u\|_{C^2} + \|v\|_{C^2})\,\|u-v\|_{W^{2,2}} + \|g-f\|_{L^2}\,. \tag{5.15}$$

Putting (5.14) and (5.15) together, we get

$$\begin{aligned}\|u-v\|_{W^{2,2}} &\le \|\Pi(u-v)\|_{W^{2,2}} + \|(u-v)^\perp\|_{W^{2,2}}\\ &\le C(\|u\|_{C^2} + \|v\|_{C^2})\,\|u-v\|_{W^{2,2}} + C\,\|g-f\|_{L^2}\,.\end{aligned}$$

Shrinking W so that $f, g \in W$ guarantees $C(\|u\|_{C^2} + \|v\|_{C^2}) < \frac{1}{2}$, this gives the required estimate.

Step 3: Establishing (5.5). Now consider the set $\widetilde{U} = \{u \in U \mid \Pi u \in W\}$. We will prove that:

(i) $\widetilde{U}$ is a neighbourhood of the origin in $C^{2,\alpha}$.

(ii) For all $u \in \widetilde{U}$ we have $\|\Psi\Pi u - \Pi u\|_{W^{2,2}} \le C\,\|\Pi u\|_{L^2}^2$.

We will then shrink U down to $\widetilde{U}$ so that (5.5) and the claim $u \in U \Rightarrow \Pi u \in W \cap K$ hold. By (i), U will still remain a neighbourhood of the origin in $C^{2,\alpha}$.

To prove (i), take any $u \in U$ and estimate

$$\|\Pi u\|_{C^{0,\alpha}} \le C\,\|\Pi u\|_{L^2} \le C\,\|u\|_{L^2} \le C\,\|u\|_{L^\infty} \le C\,\|u\|_{C^{2,\alpha}}\,. \tag{5.16}$$

The first expression makes sense since $\mathcal{K}$ contains only smooth functions (therefore admitting the $C^{0,\alpha}$ norm). The first inequality uses the equivalence of norms on $\mathcal{K}$ by finite-dimensionality, and the third inequality uses the compactness of M. Since W is a neighbourhood of 0 in $C^{0,\alpha}$, (5.16) implies that as long as $\|u\|_{C^{2,\alpha}}$ is sufficiently small, then $\Pi u \in W$. This proves (i).

To prove (ii), let $u \in \widetilde{U}$ and apply (5.4) with $f = \Pi u$ and $g = \mathcal{N}u$ to get

$$\|\Psi\Pi u - u\|_{W^{2,2}} = \|\Psi\Pi u - \Psi\mathcal{N}u\|_{W^{2,2}} \le C\,\|\Pi u - \mathcal{N}u\|_{L^2} = C\,\|\mathcal{M}u\|_{L^2}\,, \tag{5.17}$$

where the last equality is (5.12). Next, we reason exactly as in (5.16) to estimate

$$\|\Pi u\|_{C^{2,\alpha}} \le C\,\|\Pi u\|_{L^2} \le C\,\|u\|_{L^2} \le C\,\|u\|_{L^\infty} \le C\,\|u\|_{C^{2,\alpha}}\,.$$

If we shrink $\widetilde{U}$ enough, then for $u \in \widetilde{U}$ this estimate makes $\|\Pi u\|_{C^{2,\alpha}}$ small enough to ensure $\Pi u \in U$, and we also have $\Pi(\Pi u) = \Pi u \in W$. This means $\Pi u \in \widetilde{U}$, so (5.17) comes into effect with u replaced by Πu, giving

$$\|\Psi\Pi u - \Pi u\|_{W^{2,2}} \leq C \|\mathcal{M}\Pi u\|_{L^2} . \tag{5.18}$$

We need to further bound this by $C \|\Pi u\|_{L^2}^2$. Using Taylor's theorem (2.10) on $\mathcal{M}$ centred at the origin in C^2, there exists $s_* \in [0,1]$ such that

$$\mathcal{M}\Pi u = \mathcal{M}(0) + L\Pi u + \frac{1}{2} d^2\mathcal{M}|_{s_*\Pi u}(\Pi u, \Pi u) = \frac{1}{2} d^2\mathcal{M}|_{s_*\Pi u}(\Pi u, \Pi u).$$

This yields (since $\mathcal{M}$ is C^2)

$$\|\mathcal{M}\Pi u\|_{L^\infty} \leq \frac{1}{2} \left\|d^2\mathcal{M}|_{s_*\Pi u}(\Pi u, \Pi u)\right\|_{L^\infty} \leq \frac{1}{2} \sup_{s\in[0,1]} \left\|d^2\mathcal{M}|_{s\Pi u}\right\|_{\text{op}} \|\Pi u\|_{C^2}^2 \leq C \|\Pi u\|_{C^2}^2 , \tag{5.19}$$

where $\|\cdot\|_{\text{op}}$ here is the operator norm on bilinear forms $C^2 \times C^2 \to C^0$. Combining (5.18) and (5.19) then using the equivalence of norms on $\mathcal{K}$, we get

$$\|\Psi\Pi u - \Pi u\|_{W^{2,2}} \leq C \|\mathcal{M}\Pi u\|_{L^2} \leq C \|\mathcal{M}\Pi u\|_{L^\infty} \leq C \|\Pi u\|_{C^2}^2 \leq C \|\Pi u\|_{L^2}^2 ,$$

which proves (ii). Now we shrink U down to $\widetilde{U}$ as mentioned at the start of this step.

Step 4: Establishing (5.6). A consequence of (5.5) is that

$$d(\Psi \circ \Pi)|_0 = \Pi. \tag{5.20}$$

To see why, suppose $h \in L^2$ has small L^2 norm. Then $\|\Pi h\|_{L^2}$ and $\|\Pi h\|_{C^{2,\alpha}}$ are both small. Thus, for h near the origin in L^2 we get $\Pi h \in U$, so (5.5) gives

$$\begin{aligned} \|(\Psi \circ \Pi)h - (\Psi \circ \Pi)(0) - \Pi h\|_{L^2} &= \|\Psi\Pi h - \Pi h\|_{L^2} \leq \|\Psi\Pi h - \Pi h\|_{W^{2,2}} \\ &= \|\Psi\Pi(\Pi h) - \Pi(\Pi h)\|_{W^{2,2}} \leq C \|\Pi h\|_{L^2}^2 \leq C \|h\|_{L^2}^2 = o(\|h\|_{L^2}), \end{aligned}$$

which is precisely what (5.20) means.

Let $\widetilde{W} = \{\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d \mid \xi^j\varphi_j \in W \cap K\}$, and define $f : \mathbb{R}^d \to \mathbb{R}$ by $f(\xi) = \mathcal{E}(\Psi(\xi^j\varphi_j))$. Then for each $\eta \in \mathbb{R}^d$ and $\xi \in \widetilde{W}$, we have

$$\begin{aligned} \langle \eta, \nabla f(\xi)\rangle_{\mathbb{R}^d} &= \frac{d}{ds}\Big|_{s=0} f(\xi + s\eta) = \frac{d}{ds}\Big|_{s=0} \mathcal{E}(\Psi(\xi^j\varphi_j + s\eta^j\varphi_j)) \\ &= -\left\langle \mathcal{M}(\Psi(\xi^j\varphi_j)), d\Psi|_{\xi^j\varphi_j}(\eta^j\varphi_j)\right\rangle_{L^2} \\ &= -\left\langle \mathcal{M}(\Psi(\xi^j\varphi_j)), (d\Psi|_{\xi^j\varphi_j} - \Pi)(\eta^j\varphi_j)\right\rangle_{L^2} - \langle \mathcal{M}(\Psi(\xi^j\varphi_j)), \eta^j\varphi_j\rangle_{L^2} . \end{aligned} \tag{5.21}$$

Since Ψ is real-analytic, $d\Psi$ is locally Lipschitz, but we can simply restrict W (hence U) to make $d\Psi$ Lipschitz on its domain. Now (5.20) gives the estimate

$$\left\|(d\Psi|_{\xi^j\varphi_j} - \Pi)(\eta^j\varphi_j)\right\|_{L^2} = \left\|(d\Psi|_{\xi^j\varphi_j} - d\Psi|_0)(\eta^j\varphi_j)\right\|_{L^2} \leq C|\xi||\eta|. \tag{5.22}$$

The second inequality uses the fact that all p-norms are equivalent, so $|\eta|_1 \leq C|\eta|_2 = C|\eta|$. Let us now choose $\eta = \frac{\nabla f(\xi)}{|\nabla f(\xi)|}$ so that by (5.21), (5.22) and the Cauchy–Schwarz inequality,

$$|\nabla f(\xi)| \leq (1 + C|\xi|) \left\|\mathcal{M}(\Psi(\xi^j\varphi_j))\right\|_{L^2} .$$

Shrinking $\widetilde{W}$ (hence W and U accordingly) so that $C|\xi| \leq 1$, this yields (5.6). □

5.3 Proof of Theorem 5.1

We will now prove Theorem 5.1, following [Sch14]. We seem to be able to find some minor simplifications to the original proof, which hopefully makes our presentation more lucid. On the other hand, our Lemma 5.7 expands on important details which are absent in the original, and these details tie in heavily with Appendix B.

In this section, $\Sigma \subset \mathbb{R}^{n+1}$ is a fixed compact, embedded shrinker. The key step is to show that if an RMCF has Σ as a tangent flow, then it eventually becomes a small $C^{2,\alpha}$ graph over Σ forever. Uniqueness will follow easily. This is a consequence of two lemmas: the first says that a $C^{2,\alpha}$ graphical bound for RMCF can be extended in time as long as the L^2 norm remains small.

Lemma 5.6. *Given $\sigma_0 > 0$, there exists $\delta > 0$ so that if $\{\Sigma_s\}_{s\in[\tau,\infty)}$ is a family of hypersurfaces evolving by RMCF, and Σ_s is the graph over Σ for $s \in [\tau, \tau+1]$ of a smooth function $u(\cdot, s)$ (herein written $u(s)$) with*

$$\sup_{s\in[\tau,\tau+1]} \|u(s)\|_{C^{2,\alpha}} \leq \sigma_0 \quad \textit{and} \quad \sup_{s\in[\tau,\tau+1]} \|u(s)\|_{L^2} \leq \delta,$$

then Σ_s is the graph over Σ for $s \in [\tau, \tau+2]$ of an extension $\widetilde{u}$ of u in time, with

$$\sup_{s\in[\tau,\tau+2]} \|\widetilde{u}(s)\|_{C^{2,\alpha}} \leq \sigma_0.$$

Proof. If this were false, then there is a sequence of RMCFs $\{\Gamma_s^{(k)}\}_{s\in[\tau,\infty),k\in\mathbb{N}}$ so that the following holds. For each k, we have that $\Gamma_s^{(k)}$ is the graph over Σ for $s \in [\tau, \tau+1]$ of a smooth function $u^{(k)}(\cdot, s)$ with

$$\sup_{s\in[\tau,\tau+1]} \|u^{(k)}(s)\|_{C^{2,\alpha}} \leq \sigma_0 \quad \text{and} \quad \sup_{s\in[\tau,\tau+1]} \|u^{(k)}(s)\|_{L^2} \leq \frac{1}{k}, \tag{5.23}$$

but there is no extension $\widetilde{u}^{(k)}$ of $u^{(k)}$ to time $s \in [\tau, \tau+2]$ so that $\Gamma_s^{(k)}$ is the graph of $\widetilde{u}^{(k)}$ with

$$\sup_{s\in[\tau,\tau+2]} \|\widetilde{u}^{(k)}(s)\|_{C^{2,\alpha}} \leq \sigma_0. \tag{5.24}$$

By the compactness theorem for rescaled Brakke flows (see §3.3.3), there is a subsequence $\Gamma_s^{(k_i)}$, $k_i \to \infty$, converging to another rescaled Brakke flow Γ in a suitable Brakke sense. We argue that the convergence is in fact smooth. To see this, note that each $u^{(k)}$ satisfies a parabolic equation by Proposition B.3. We may therefore use parabolic bootstrapping to turn the uniform $C^{2,\alpha}$ bound of (5.23) into uniform $C^{\ell,\alpha}$ bounds for each $\ell \in \mathbb{N}$, say

$$\sup_{s\in[\tau,\tau+1]} \|u^{(k)}(s)\|_{C^{\ell,\alpha}} \leq \sigma_\ell. \tag{5.25}$$

for each k. By (5.23), we have $u^{(k_i)}(\cdot, s)|_{s\in[\tau,\tau+1]} \to 0$ in $L^2(\Sigma \times [\tau, \tau+1])$. Coupled with (5.25), this implies the convergence also holds in $C^{\ell-1}(\Sigma \times [\tau, \tau+1])$ for each ℓ, and therefore in C^∞.

It follows that the limit rescaled Brakke flow Γ is actually a smooth RMCF coinciding with the static RMCF Σ during the time interval $[\tau, \tau+1]$. But this implies Γ coincides with Σ forever by the uniqueness of RMCF (this is the rescaled version of Theorem 3.5). Since every timeslice of Γ is smooth with unit multiplicity, Brakke's regularity theorem [Bra78] implies the convergence $\Gamma_s^{(k_i)} \to \Gamma$ is smooth on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$. This contradicts (5.24). □

The second lemma uses the Łojasiewicz–Simon gradient inequality to bound the L^2 change over time for a graphical RMCF over Σ. This upper bound becomes arbitrarily small for large enough times, so it is later used to realise the L^2 condition of Lemma 5.6.

Lemma 5.7. *There exist $\sigma_0 > 0$, $\theta \in (0, \frac{1}{2})$ and $C > 0$ depending on n so that if Σ_s is an RMCF in $\mathbb{R}^{n+1}$ for $s \in [\tau_1, \tau_2]$, and Σ_s is the graph over Σ for each s of a smooth function $u(\cdot, s)$ with*

$$\sup_{s\in[\tau_1,\tau_2]} \|u(s)\|_{C^{2,\alpha}} \le \sigma_0,$$

then

$$\sup_{s\in[\tau_1,\tau_2]} \|u(s) - u(\tau_1)\|_{L^2} \le \int_{\tau_1}^{\tau_2} \left\|\frac{\partial u}{\partial s}\right\|_{L^2} ds \le \frac{C}{\theta}(\mathcal{F}_\Sigma(u(\tau_1)) - \mathcal{F}_\Sigma(0))^\theta. \tag{5.26}$$

Proof. The first inequality in (5.26) comes from

$$\sup_{s\in[\tau_1,\tau_2]} \|u(s) - u(\tau_1)\|_{L^2} = \sup_{s\in[\tau_1,\tau_2]} \left\|\int_{\tau_1}^{s} \frac{\partial u}{\partial \tau} d\tau\right\|_{L^2} \le \sup_{s\in[\tau_1,\tau_2]} \int_{\tau_1}^{s} \left\|\frac{\partial u}{\partial \tau}\right\|_{L^2} d\tau \le \int_{\tau_1}^{\tau_2} \left\|\frac{\partial u}{\partial \tau}\right\|_{L^2} d\tau,$$

so it remains to prove the second inequality. By the explicit expression (B.1) for $\mathcal{F}_\Sigma$, we have

$$\mathcal{F}_\Sigma(u) = \int_\Sigma (4\pi)^{-\frac{n}{2}} e^{-\frac{|x+u(x)\mathbf{n}(x)|^2}{4}} \nu_u(x),$$

where $\nu_u(x)$ is the relative volume element. By Lemma B.1, ν_u can be written as

$$\nu_u(x) = \nu(x, u(x), \nabla u(x)),$$

where ν is analytic in the u and ∇u entries for $\|u\|_{C^0}$ sufficiently small, and is uniformly convex in the ∇u entry for $\|u\|_{C^1}$ sufficiently small. Therefore, $\mathcal{F}_\Sigma$ satisfies Assumption 5.2. Let $\mathcal{M}_\Sigma$ be its Euler-Lagrange functional. Then Theorem 5.3 gives $\sigma_0, C > 0$ and $\beta \in (\frac{1}{2}, 1)$ depending on n and the form of $\mathcal{F}_\Sigma$ (thus only on n as $\mathcal{F}_\Sigma$ is otherwise fixed) such that for all $u \in C^{2,\alpha} \cap B_{\sigma_0}(0)$,

$$\|\mathcal{M}_\Sigma(u)\|_{L^2} \ge C|\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(0)|^\beta. \tag{5.27}$$

Write $\rho(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$, and let x and y be generic points on Σ and Σ_s respectively, related bijectively by $y = x + u(x, s)\mathbf{n}(x)$. By Lemma 3.24, we have

$$\begin{aligned} \frac{d}{ds}\mathcal{F}_\Sigma(u(s)) &= -\int_{\Sigma_s} \left(H^{\Sigma_s} - \frac{\langle y, \mathbf{n}^{\Sigma_s}\rangle}{2}\right)^2 \rho(y) \\ &= -\int_\Sigma \left(H_{u(s)} - \frac{\langle x + u(x,s)\mathbf{n}(x), \mathbf{n}_{u(s)}\rangle}{2}\right)^2 \rho(y)\nu_{u(s)}(x), \end{aligned} \tag{5.28}$$

where $H_{u(s)}(x) = H^{\Sigma_s}(y)$ and $\mathbf{n}_{u(s)}(x) = \mathbf{n}^{\Sigma_s}(y)$. To simplify notation, let X be the expression

$$X = \int_\Sigma \left(H_{u(s)} - \frac{\langle x + u(x,s)\mathbf{n}(x), \mathbf{n}_{u(s)}\rangle}{2}\right)^2 \rho(y)\nu_{u(s)}(x).$$

By the Cauchy–Schwarz inequality and Proposition B.2,

$$\begin{aligned} X &\geq \int_\Sigma \left(H_{u(s)} - \frac{\langle x + u(x,s)\mathbf{n}(x), \mathbf{n}_{u(s)} \rangle}{2} \right)^2 \langle \mathbf{n}, \mathbf{n}_u \rangle^2 \rho(y) \nu_{u(s)}(x) \\ &= \int_\Sigma |\mathcal{M}_\Sigma(u(s))|^2 (\rho(y)\nu_{u(s)}(x))^{-1}. \end{aligned} \tag{5.29}$$

Since $\nu_u = 1$ when $u = 0$, and ν depends smoothly on u and ∇u, we can use the assumption that $\|u(s)\|_{C^{2,\alpha}} \leq \sigma_0$ to bound $\nu_{u(s)}$ from above by a constant depending on σ_0. Using this in (5.29) together with $\rho(y)^{-1} \geq (4\pi)^{\frac{n}{2}}$, we further estimate

$$X \geq C \int_\Sigma |\mathcal{M}_\Sigma(u(s))|^2 = C \, \|\mathcal{M}_\Sigma(u(s))\|_{L^2}^2 \,, \tag{5.30}$$

where $C = C(n, \sigma_0) = C(n) > 0$. At the same time, using the evolution equation for RMCF (Lemma 3.21) followed by Corollary B.4, there exists $C = C(n)$ such that

$$X = \int_\Sigma \left\langle \frac{\partial u}{\partial s}\mathbf{n}, \mathbf{n}_{u(s)} \right\rangle^2 \rho(y)\nu_{u(s)}(x) \geq C \int_\Sigma \left| \frac{\partial u}{\partial s} \right|^2 \rho(y)\nu_{u(s)}(x).$$

Since Σ is compact and $\|u\|_{C^{2,\alpha}} \leq \sigma_0$, it follows that $|y| = |x + u(x)\mathbf{n}(x)|$ is bounded and hence $\rho(y) \geq C$ for some positive $C = C(n, \sigma_0) = C(n)$. Like above, we can bound $\nu_{u(s)}$ from below by $C = C(\sigma_0) = C(n)$. Therefore

$$X \geq C \int_\Sigma \left| \frac{\partial u}{\partial s} \right|^2 = C \left\| \frac{\partial u}{\partial s} \right\|_{L^2}^2 . \tag{5.31}$$

Putting (5.30) and (5.31) back into (5.28), then using (5.27), we get

$$\begin{aligned} \frac{d}{ds}\mathcal{F}_\Sigma(u(s)) = -X &\leq -C \, \|\mathcal{M}_\Sigma(u(s))\|_{L^2} \left\| \frac{\partial u}{\partial s} \right\|_{L^2} \\ &\leq -C |\mathcal{F}_\Sigma(u(s)) - \mathcal{F}_\Sigma(0)|^\beta \left\| \frac{\partial u}{\partial s} \right\|_{L^2} . \end{aligned}$$

We therefore have

$$\begin{aligned} -\frac{d}{ds}(\mathcal{F}_\Sigma(u(s)) - \mathcal{F}_\Sigma(0))^{1-\beta} &= -(1-\beta)(\mathcal{F}_\Sigma(u(s)) - \mathcal{F}_\Sigma(0))^{-\beta} \left(\frac{d}{ds}\mathcal{F}_\Sigma(u(s)) \right) \\ &\geq C(1-\beta) \left\| \frac{\partial u}{\partial s} \right\|_{L^2} . \end{aligned} \tag{5.32}$$

Setting $\theta = 1 - \beta$ then integrating (5.32) yields the second inequality of (5.26). □

Proof of Theorem 5.1. Let Σ_s be an RMCF associated to an MCF of compact, embedded hypersurfaces in $\mathbb{R}^{n+1}$. That is, Σ_s is obtained from an MCF by means of Definition 3.20, but we now refer to the hypersurfaces themselves instead of the parametrising maps, seeing as they are embedded. Suppose Σ arises as a tangent flow of Σ_s. We need to show that Σ is the unique tangent flow.

Let $\sigma_0 = \sigma_0(n)$ be given by Lemma 5.7, and $\delta = \delta(\sigma_0)$ be given by Lemma 5.6. Further choose $0 < \sigma < \sigma_0$ so that whenever $u \in C^{2,\alpha}(\Sigma)$ satisfies $\|u\|_{C^{2,\alpha}} \leq \sigma$, we have

(i) $\|u\|_{L^2} \le \frac{\delta}{2}$;

(ii) $\frac{C}{\theta}|\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(0)|^\theta \le \frac{\delta}{2}$, where C and θ are given by Lemma 5.7 depending on n.

The inequality (i) is possible since $\|u\|_{L^2} \le C\,\|u\|_{C^{2,\alpha}}$ where C depends on the (finite) volume of Σ. Meanwhile, (ii) is possible since $\mathcal{F}_\Sigma(u) \approx \mathcal{F}_\Sigma(0)$ if $\|u\|_{C^1}$ is small (see §B.2).

Suppose Σ_s is an RMCF which has Σ as a tangent flow. Then there is a sequence of times $s_i \to \infty$ such that the sequence of RMCFs $\{\Sigma_s\}_{s\in[s_i,s_i+1]}$ converges smoothly to the RMCF of Σ, which is stationary.[2] This means we can find a time s_* such that Σ_s is a normal graph over Σ for all $s \in [s_*, s_*+1]$, and the graph function $u(\cdot, s)$ satisfies $\|u(s)\|_{C^{2,\alpha}} \le \sigma$. So (i) and (ii) apply to $u(s)$ for $s \in [s_*, s_*+1]$. By (i) and Lemma 5.6, Σ_s is the graph of $\widetilde{u}(s)$ over Σ for $s \in [s_*, s_*+2]$, and $\|\widetilde{u}(s)\|_{C^{2,\alpha}} \le \sigma_0$. Lemma 5.7 now implies that for $s \in [s_*+1, s_*+2]$,

$$\|\widetilde{u}(s)\|_{L^2} \le \|\widetilde{u}(s_*+1)\|_{L^2} + \frac{C}{\theta}(\mathcal{F}_\Sigma(\widetilde{u}(s_*+1)) - \mathcal{F}_\Sigma(0))^\theta \le \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

The hypotheses of Lemma 5.6 are now met by $\widetilde{u}(s)$ for $s \in [s_*, s_*+2]$. It follows that for $s \in [s_*, s_*+3]$, Σ_s is the normal graph over Σ of $\widetilde{u}$ with $\|\widetilde{u}(s)\|_{C^{2,\alpha}} \le \sigma_0$. Applying Lemma 5.7 again, we get that for $s \in [s_*+2, s_*+3]$,

$$\|\widetilde{u}(s)\|_{L^2} \le \|\widetilde{u}(s_*+1)\|_{L^2} + \frac{C}{\theta}(\mathcal{F}_\Sigma(\widetilde{u}(s_*+1)) - \mathcal{F}_\Sigma(0))^\theta \le \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Iterating indefinitely gives that Σ_s is the graph of $\widetilde{u}(s)$ over Σ for $s \in [s_*, \infty)$, and

$$\|\widetilde{u}(s)\|_{C^{2,\alpha}} \le \sigma_0. \tag{5.33}$$

Then whenever i and s are such that $s \ge s_i \ge s_*$, Lemma 5.7 gives

$$\|\widetilde{u}(s)\|_{L^2} \le \|\widetilde{u}(s_i)\|_{L^2} + \frac{C}{\theta}(\mathcal{F}_\Sigma(\widetilde{u}(s_i)) - \mathcal{F}_\Sigma(0))^\theta. \tag{5.34}$$

As $i \to \infty$, we have $\widetilde{u}(s_i) \to 0$ uniformly since $\Sigma_{s_i} \to \Sigma$ smoothly by how s_i was chosen. Then both terms on the right of (5.34) approach zero, giving

$$\lim_{s\to\infty} \|\widetilde{u}(s)\|_{L^2} = 0,$$

i.e. $\widetilde{u}(s) \to 0$ in L^2. Since $\widetilde{u}$ is a graphical RMCF, it obeys a parabolic equation (Lemma B.3). Parabolic bootstrapping turns the uniform $C^{2,\alpha}$ bound (5.33) into $C^{k,\alpha}$ bounds for all k, say

$$\|\widetilde{u}(s)\|_{C^{k,\alpha}} \le C_k$$

for all $s \in [s_*, \infty)$. However, L^2 convergence and uniform boundedness in $C^{k,\alpha}$ implies C^{k-1} convergence, so $\widetilde{u}(s) \to 0$ in C^k for all k. We are done, because this shows that the hypersurfaces $\Sigma_s = \mathrm{graph}_\Sigma(\widetilde{u}(s))$ of the RMCF converge in C^∞ to Σ along every sequence of times $s_i \to \infty$. □

[2]This appeals to the perspective of tangent flows as rescaling limits of MCFs; see the end of §3.3.3.

Chapter 6

Uniqueness of Cylindrical Tangent Flows

To prove uniqueness of tangent flows for all mean convex mean curvature flows (Theorem 3.31), it remains to prove that all cylindrical blowups are unique:

Theorem 6.1 ([CM15]). *If a unit multiplicity cylinder arises as a tangent flow of a compact, embedded MCF, then it is the unique tangent flow at that point.*

Like uniqueness of compact tangent flows, the proof is driven by Łojasiewicz-type inequalities. However, the Łojasiewicz–Simon gradient inequality of Theorem 5.3 is unusable, in part because the tangent flow in question is now noncompact, but also because the compact hypersurfaces of the RMCF are never graphs over the whole cylinder. In this chapter, we will follow Colding and Minicozzi's paper [CM15] where new Łojasiewicz-type inequalities are developed, using entirely novel techniques, leading to a successful proof of Theorem 6.1. Analogues of both inequalities (5.1), (5.2) will be proved, with the former implying the latter.

We start with a synopsis to distill the key ideas of the proof. This summary is our own, and we hope it illuminates an otherwise highly technical chapter. For the main matter, our treatment retains the skeleton of the original paper, but our statements and proofs diverge from the original at numerous points (most notably in §6.2 and §6.5, and to an extent §6.3). These are our efforts to improve accuracy and intelligibility, and to correct any mistakes we could find. These changes typically blend our own arguments with those from various other sources (e.g. [Man14], [CM19b], [Zhu20]).

We tend to omit proofs that are already detailed enough in the paper and where there is little to add, chiefly those in Section 3 of the paper. We simply state the results and move on. In exchange, the proofs we elect to include are presented more meticulously than in the original.

6.0.1 Preliminaries for this chapter

In this chapter, all hypersurfaces are embedded, and we use the weighted L^p norms from §4.1. Thus, for a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, we have $\|u\|^p_{L^p(\Sigma)} = \int_\Sigma |u|^p e^{-\frac{|x|^2}{4}}$. Similarly we define

$\|u\|_{L^p(B_R\cap\Sigma)}$ for $R>0$. We write $\|u\|_{L^p}$ and $\|u\|_{L^p(B_R)}$ if there is an obvious candidate for Σ.

Two important quantities on Σ are the tensor τ and the function $\phi:\Sigma\to\mathbb{R}$, defined by

$$\tau=\frac{A}{H},\qquad \phi(x)=-H(x)+\frac{\langle x,\mathbf{n}(x)\rangle}{2}.$$

Note that $\nabla\tau=0$ on a cylinder, and $\phi\equiv 0$ if and only if Σ is a shrinker (Theorem 3.9). Next, we define the *entropy* of Σ. For each $x_0\in\mathbb{R}^{n+1}$ and $\tau>0$, let

$$\mathcal{F}_{x_0,\tau}(\Sigma)=(4\pi\tau)^{-\frac{n}{2}}\int_\Sigma e^{-\frac{|x-x_0|^2}{4\tau}},$$

so that $\mathcal{F}_{0,1}=\mathcal{F}$. The entropy of Σ is defined by

$$\lambda(\Sigma)=\sup_{x_0\in\mathbb{R}^{n+1},\tau>0}\mathcal{F}_{x_0,\tau}(\Sigma). \tag{6.1}$$

Arguing similarly as (4.1), polynomial volume growth implies finite entropy $\lambda(\Sigma)\le\lambda_0<\infty$. If Σ is a shrinker, then the converse holds: bounded entropy implies $\mathcal{H}^n(B_R(x_0)\cap\Sigma)\le C(\lambda_0)R^n$ for all $x_0\in\mathbb{R}^{n+1}$ [CZ13]. Bounded entropy also yields a cutoff lemma:

Lemma 6.2 ([CM19b]). *If $\Sigma\subset\mathbb{R}^{n+1}$ is a hypersurface with $\lambda(\Sigma)\le\lambda_0<\infty$, then for any Euclidean ball $B_R(x_0)\subset\mathbb{R}^{n+1}$ we have*

$$\int_{\Sigma\setminus B_R(x_0)}|x-x_0|^m e^{-\frac{|x-x_0|^2}{4}}\le C\lambda_0R^\rho e^{-\frac{R^2}{4}}<\infty,$$

where $C=C(m,n,\lambda_0)$ and $\rho=\rho(m,n)$.

In an earlier paper, uniqueness of type for cylindrical tangent flows was proved:

Theorem 6.3 ([CIM15]). *If a unit multiplicity cylinder arises as a tangent flow of the RMCF at a point of a compact, embedded MCF, then all tangent flows are rotations of this cylinder.*

Theorem 6.1 strengthens this by ruling out the rotational freedom. The proof uses Theorem 6.3 directly. We will not prove Theorem 6.3; although this may seem an excessive leap, many proofs in this chapter are borne out of ideas from [CIM15], so readers will find that paper accessible after reading this chapter or the paper [CM15] which this chapter is based on.

6.1 Synopsis

Let us now outline how Theorem 6.1 will be proved. Figure 6.1 illustrates the pipeline of main ingredients in the proof, which we will run through shortly.

6.1.1 The cylindrical scale

Let $\mathcal{C}_k$ be the set of all rotations of $S^k_{\sqrt{2k}}\times\mathbb{R}^{n-k}$ about the origin in $\mathbb{R}^{n+1}$, where $k\in\{1,\dots,n-1\}$. If a tangent flow is cylindrical, then it belongs to $\mathcal{C}_k$ for some k (Theorem 4.1), and in fact all tangent flows belong to $\mathcal{C}_k$ (Theorem 6.3). The Łojasiewicz inequalities used to prove Theorem 6.1 are therefore designed for hypersurfaces already 'close' to $\mathcal{C}_k$. To quantify this closeness, we will use a *cylindrical scale* defined for an embedded hypersurface $\Sigma\subset\mathbb{R}^{n+1}$ as follows.

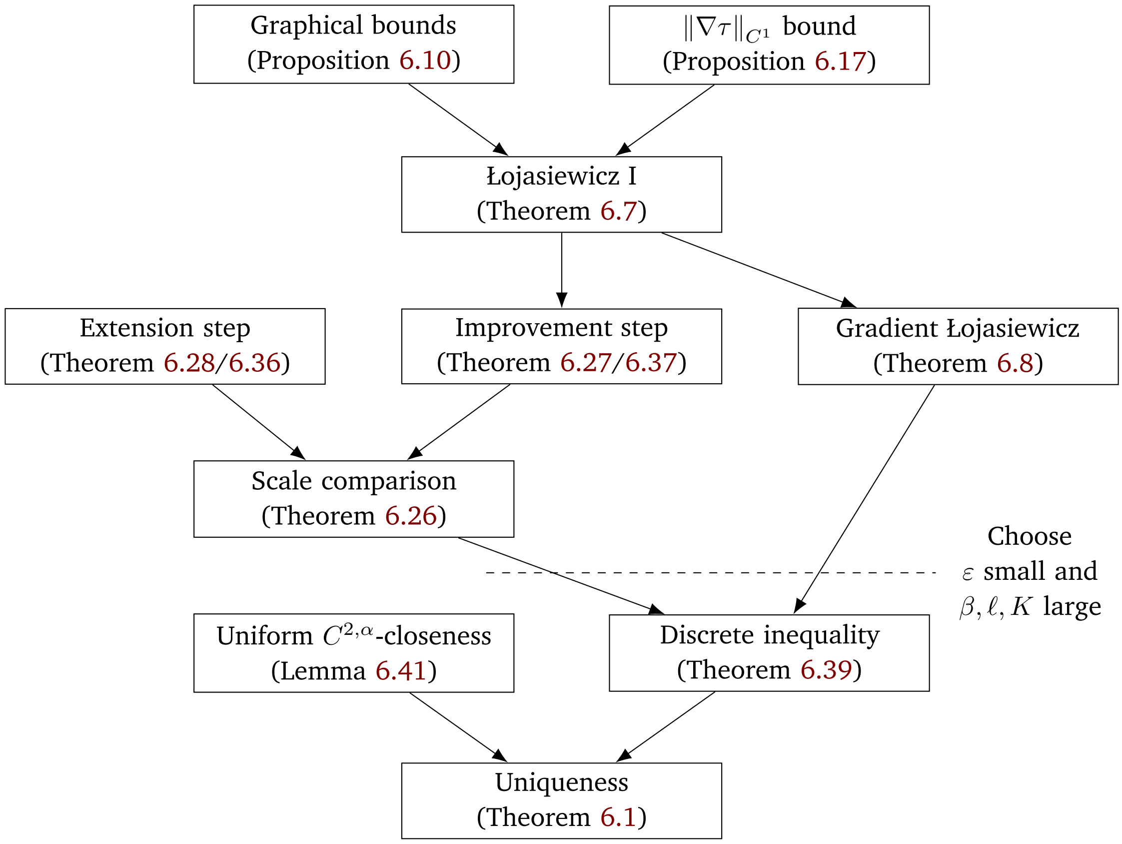

Figure 6.1: Main theorems in the proof of Theorem 6.1 and their relationships.

Definition 6.4. Given $\varepsilon, R > 0$, we say that Σ is $(\varepsilon, R, C^{2,\alpha})$-*close* to another hypersurface Γ if $B_R \cap \Sigma$ is the normal graph of some $u \in C^{2,\alpha}(\Gamma)$ over Γ with $\|u\|_{C^{2,\alpha}} \leq \varepsilon$.

Definition 6.5. Given $\varepsilon > 0$, $\ell \in \mathbb{N}$ and $K > 0$, the *cylindrical scale* $\mathbf{r}_{\varepsilon,\ell,K}(\Sigma)$ is the maximum R such that

- Σ is $(\varepsilon, R, C^{2,\alpha})$-close to a cylinder in $\mathcal{C}_k$ for some k, and $|\nabla^\ell A| \leq K$ on $B_R \cap \Sigma$.

As $\mathbf{r}_{\varepsilon,\ell,K}(\Sigma)$ depends on ε, ℓ and K, this really defines a family of cylindrical scales. For a fixed K, taking ε small and ℓ large makes $B_R \cap \Sigma$ increasingly cylinder-like whenever $R \leq \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)$. We think of $B_R \cap \Sigma$ as being close to $\mathcal{C}_k$ (this qualification of course depends on ε, ℓ and K).

Lemma 6.6. *Let $R \leq \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)$. On $B_R \cap \Sigma$, there is a lower bound for H depending on ε, and for each $j \leq \ell$ there are upper bounds for $|\nabla^j A|, |\nabla^j H|, |\nabla^j \tau|, |\nabla^j \phi|$ depending on ε, j, K.*

Proof. Suppose $B_R \cap \Sigma$ is the graph of u over $\Gamma \in \mathcal{C}_k$ with $\|u\|_{C^{2,\alpha}} \leq \varepsilon$. Since Γ has constant $|A|^2 = \frac{1}{2}$ and $H = \sqrt{k/2}$, and the A and H of $B_R \cap \Sigma$ depend on second derivatives of u, we get control over $|A|$ and a lower bound for H using ε. The bounds on $|A|$ and $|\nabla^\ell A|$ interpolate to bounds on $|\nabla^j A|$ for all $j \leq \ell$, and therefore bounds on $|\nabla^j H|$. Since $\nabla^j \tau$ is an expression in A, H and their covariant derivatives with only H appearing the denominator, it is also bounded. Finally, ϕ depends on up to second derivatives of u by Lemma B.1 (specifically $\phi = \frac{1}{2}\eta_u - H_u$ over there). But the $|\nabla^j A|$ bounds control the $(j+2)$-th derivatives of u, so we can use this to control $|\nabla^j \phi|$ for all $j \leq \ell$. ∎

6.1.2 Łojasiewicz inequalities for cylinder-like hypersurfaces

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface and $\Gamma \in \mathcal{C}_k$. For $x \in \Sigma$, let $w_\Gamma(x)$ be the distance from x to the axis of Γ. Using this, we define a weighted L^2 distance from Σ to $\mathcal{C}_k$ within B_R by

$$d_{\mathcal{C}_k}(B_R \cap \Sigma)^2 = \inf_{\Gamma \in \mathcal{C}_k} \left\| w_\Gamma - \sqrt{2k} \right\|^2_{L^2(B_R \cap \Sigma)} = \inf_{\Gamma \in \mathcal{C}_k} \int_{B_R \cap \Sigma} (w_\Gamma - \sqrt{2k})^2 e^{-\frac{|x|^2}{4}}.$$

The first Łojasiewicz inequality bounds this distance as long as $B_R \cap \Sigma$ is sufficiently cylinder-like. This translates to the requirement that $R \leq \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)$ where ε is small and ℓ is large. We also need a lower bound $R \geq R_0$ to make sure everything happens on a nontrivial scale to begin with.

Theorem 6.7 (Łojasiewicz I, 0.24[1]). *Given n, there exist ε_0 and ℓ_0 with the following property. For all $\lambda_0 > 0$, $\varepsilon \leq \varepsilon_0$, $\ell \geq \ell_0$ and $K > 0$, there exists $R_0 = R_0(n, \lambda_0, \varepsilon, \ell, K)$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface with $\lambda(\Sigma) \leq \lambda_0$ and $R \in [R_0, \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)]$, then*

$$d_{\mathcal{C}_k}(B_R \cap \Sigma)^2 \leq C R^\rho \left\{ \|\phi\|^{d_{\ell,n}}_{L^1(B_R)} + e^{-\frac{d_{\ell,n} R^2}{4}} \right\},$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$, $\rho = \rho(n)$ and $d_{\ell,n} \in (0,1) \nearrow 1$ as $\ell \to \infty$.

Since every $\Gamma \in \mathcal{C}_k$ is a shrinker and therefore has $\phi \equiv 0$, this bounds the distance from $B_R \cap \Sigma$ to the zero set of ϕ using ϕ itself, in similar spirit to (5.1). Meanwhile, there is an error term coming from a cutoff as Σ is not an entire graph over the cylinder. The proof is detailed in §6.2 and §6.3, and uses two delicate results: a graphical proposition and a bound for $\|\nabla\tau\|_{C^1}$.

Colding and Minicozzi's original statement does not impose a variable lower bound $R \geq R_0$, but we believe it is needed for technical reasons in the proof. Anyway, the lower bound is on the whole harmless, as we are only interested in using the theorem when $R \approx \mathbf{r}_{\varepsilon,\ell,K}(\Sigma) \gg R_0$. Still, it has a spillover effect on the theorems that follow.

In §6.4, we will use Theorem 6.7 to prove a gradient inequality which generalises (5.2). Note that all cylinders in $\mathcal{C}_k$ have the same $\mathcal{F}$ value by symmetry, so we can make sense of $\mathcal{F}(\mathcal{C}_k)$. We again highlight the presence of noncompact error terms.

Theorem 6.8 (Gradient Łojasiewicz, 0.26). *Given n, there exist ε_0 and ℓ_0 with the following property. For all $\lambda_0 > 0$, $\varepsilon \leq \varepsilon_0$, $\ell \geq \ell_0$ and $K > 0$, there exists $R_0 = R_0(n, \lambda_0, \varepsilon, \ell, K)$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface with $\lambda(\Sigma) \leq \lambda_0$, $R \in [R_0, \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)]$, and $\beta \in [0,1)$, then*

$$|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)| \leq C R^\rho \left\{ \|\phi\|^{d_{\ell,n} \frac{3+\beta}{2+2\beta}}_{L^2(B_R)} + e^{-\frac{d_{\ell,n}(3+\beta) R^2}{8(1+\beta)}} + e^{-\frac{(3+\beta)(R-1)^2}{16}} \right\},$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$, $\rho = \rho(n)$ and $d_{\ell,n} \in (0,1) \nearrow 1$ as $\ell \to \infty$.

While these theorems look complicated, we emphasise that most of their conditions simply express the requirement that $B_R \cap \Sigma$ is cylinder-like. It is for this reason that we refer to them as *Łojasiewicz inequalities for cylinder-like hypersurfaces*.

[1]In this chapter, numbers in parentheses after theorem/lemma numbers indicate the closest (but often not exact) match in the numbering of the original paper [CM15].

6.1.3 From Łojasiewicz inequalities to uniqueness

The gradient inequality of Theorem 6.8 will be applied to the timeslices Σ_T of a given RMCF, then used to prove uniqueness of cylindrical tangent flows. The key milestone will be to bound each term on the right by a power greater than $\frac{1}{2}$ of $\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1})$, so that

$$|\mathcal{F}(\Sigma_T) - \mathcal{F}(\mathcal{C}_k)| \leq C(\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}))^{\frac{1+\mu}{2}} \tag{6.2}$$

for some $\mu > 0$. This is Theorem 6.39, and we call (6.2) the *discrete differential inequality*.

To turn the gradient inequality into (6.2), note that the former currently has six degrees of freedom: $n, \lambda_0, \varepsilon, \ell, K$ and β. As we are dealing with a given RMCF, n and λ_0 are enforced upon us, leaving four free variables at our disposal. The plan is to apply the gradient inequality with a judicious choice of these parameters so that (6.2) follows; there are two parts to this.

(1) We will choose β close to one and ℓ large to make the exponent on $\|\phi\|_{L^2}$ in the gradient inequality greater than one. A mean value inequality will bound $\|\phi\|_{L^2}^2$ by a constant times $\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1})$.

(2) We will choose K large to bound both error terms by a power greater than $\frac{1}{2}$ of $\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1})$. This will require a $C^{2,\alpha}$-closeness criterion to be met (Theorem 6.26).

Since ε plays no part here, we can fix it to be the $\varepsilon_0 = \varepsilon_0(n)$ of Theorem 6.8. To sketch how (2) works, observe that (1) makes the error terms essentially $e^{-\frac{R^2}{4}}$. To bound these, we ask if there exists R within the applicable range of the gradient inequality (i.e. $R \in [R_0, \mathbf{r}_{\varepsilon_0,\ell,K}(\Sigma_T)]$) satisfying

$$e^{-\frac{R^2}{4}} \leq (\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}))^{\frac{1+\mu}{2}}$$

for some $\mu > 0$. Phrasing this question differently, we define a *shrinker scale* $R(\Sigma_T)$ by

$$e^{-\frac{R(\Sigma_T)^2}{4}} = (\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}))^{\frac{1}{2}},$$

and ask whether there exists $\mu > 0$ such that

$$\mathbf{r}_{\varepsilon_0,\ell,K}(\Sigma_T) \geq (1+\mu)R(\Sigma_T).$$

In Theorem 6.26, we will show that this inequality holds when the RMCF Σ_s meets a uniform $C^{2,\alpha}$-closeness criterion and K is large. The $C^{2,\alpha}$-closeness criterion reads:

($\#_T$) For each $s \in [T - \frac{1}{2}, T+1]$, Σ_s is $(\varepsilon_E, R_1, C^{2,\alpha})$-close to a cylinder,

where ε_E and R_1 are stipulated by the theorem. Theorem 6.26 is proved by repeated iteration of two theorems, called the extension and improvement steps. The improvement step is a consequence of the first Łojasiewicz inequality, while the extension step is derived from standard regularity results for MCF. This procedure is developed in §6.5.

Thus, ($\#_T$) is the condition required to bound the error terms as per (2). Theorem 6.39 puts everything together, giving that if ($\#_T$) holds, then the discrete differential inequality (6.2) can be obtained from the gradient inequality by choosing β, ℓ and K according to (1) and (2).[2]

[2]We also need to choose β and ℓ slightly larger to absorb the R^ρ term in the gradient inequality.

The point is that if Σ_s is an RMCF with a cylindrical tangent flow, then $(\#_s)$ holds for all large s (Lemma 6.41). From the last paragraph, this means (6.2) holds on Σ_s for all large s. We will use this to show that

$$\int_1^\infty \|\phi\|_{L^1(\Sigma_s)}\, ds < \infty.$$

Since ϕ is the speed of the RMCF by Lemma 3.21, this shows that the flow has finite length, so Σ_s has a unique limit (Lemma B.5 makes this precise). Theorem 6.1 follows. Observe that this last part is basically what was outlined in §5.1.

Remark 6.9. In practice, steps (1) and (2) above are reversed. Namely, Theorem 6.26 will show that there exists such a K for each ℓ, and then β and ℓ are chosen large afterwards. This is because we actually have to choose β and ℓ depending on a constant coming out of Theorem 6.26 as the error terms are not exactly $e^{-\frac{R^2}{4}}$. The mean value inequality mentioned in (1) is also packaged into Theorem 6.26, purely for convenience. We ignored these matters in the synopsis to simplify discussion, but the essence of the strategy remains unchanged.

6.2 A graphical proposition

Over the next two sections, we will prove the first Łojasiewicz inequality, Theorem 6.7. This hinges on the next proposition, which says that if a hypersurface Σ is almost cylindrical on a small scale and $\|\phi\|_{C^1}, \|\nabla\tau\|_{C^1}$ almost vanish on a large scale, then Σ is almost cylindrical on the large scale. The proposition supplies pointwise graphical bounds to be used later.

Proposition 6.10 (2.1)**.** *Given n and K_1, there exist $\varepsilon_0 = \varepsilon_0(n)$ and $\varepsilon_1 = \varepsilon_1(n, K_1)$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface with*

(1) $H \geq \frac{1}{2}$ and $|A| + |\nabla A| \leq K_1$ on $B_R \cap \Sigma$;

(2) Σ is $(\varepsilon_0, 5\sqrt{2n}, C^2)$-close to a cylinder in $\mathcal{C}_k$ for some $k \geq 1$;

then whenever $r \in (5\sqrt{2n}, R)$ has

(3) $r\,\|\phi\|_{C^1(B_{5\sqrt{2n}})} + r^5\,\|\nabla\tau\|_{C^1(B_r)} \leq \varepsilon_1$;

(4) $B_\rho \cap \Sigma$ is connected for all $\rho \in (5\sqrt{2n}, r)$;

we have that $B_r \cap \Sigma$ is the graph over a (possibly different) cylinder in $\mathcal{C}_k$ of u with the bound

$$|u(x)| + |\nabla u(x)| \leq C\left(r\,\|\phi\|_{C^1(B_{5\sqrt{2n}})} + r^5\,\|\nabla\tau\|_{C^1(B_{|x|})}\right),$$

where $C = C(n, K_1)$.

Remark 6.11. The statement in the original paper does not have condition (4), but it seems necessary (see Step 5 of the proof in §6.2.2). Anyway, when the proposition is applied we will already assume $B_R \cap \Sigma$ is graphical over a cylinder, and this geometrically implies (4).

6.2.1 Ingredients for the proof of Proposition 6.10

The proof of Proposition 6.10 uses three lemmas. We omit proofs of the first two, instead focusing on the third where our version differs from [CM15]. The first lemma is from [CIM15]:

Lemma 6.12 ([CIM15, Corollary 4.22]). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface with $H \geq \frac{1}{2}$ and $|\nabla\tau| + |\nabla^2\tau| \leq \varepsilon < 1$. Suppose τ has at least two distinct eigenvalues $\kappa_1 \neq \kappa_2$ at $x \in \Sigma$. Then*

$$|\kappa_1\kappa_2| \leq 8\varepsilon\left(\frac{1}{|\kappa_1 - \kappa_2|} + \frac{1}{|\kappa_1 - \kappa_2|^2}\right).$$

The second lemma we need is a spherical variant of Theorem 6.7.

Lemma 6.13 (2.5). *Given k and $\alpha > 0$, there exist ε_0 and C so that if $\Sigma_0 \subset \mathbb{R}^{k+1}$ is the graph over $S^k_{\sqrt{2k}}$ of a function u satisfying $\|u\|_{C^{2,\alpha}} \leq \varepsilon_0$, then*

$$\|u\|_{C^{2,\alpha}(S^k_{\sqrt{2k}})} \leq C\,\|\phi\|_{C^{0,\alpha}(\Sigma_0)}\,.$$

The third lemma says that if Σ is almost a shrinker and is almost translation invariant in $n-k$ directions, then slicing Σ orthogonally to these directions gives a k-dimensional almost-shrinker. We will use this to slice an almost-cylinder down to an almost-sphere.

Lemma 6.14 (2.11). *Let $(x_1, \ldots, x_{n+1})$ denote coordinates in $\mathbb{R}^{n+1}$. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface, $\Pi = \{x_{k+2} = x_{k+3} = \ldots = x_{n+1} = 0\}$, and $\Sigma_0 = \Sigma \cap \Pi$. For $i \in \{k+2, k+3, \ldots, n+1\}$, define $v_i = \nabla x_i$, the tangential projection of ∂_i onto Σ. If $x \in \Sigma_0$ is a point where Σ intersects Π transversely, and for all i we have*

$$|v_i(x)| \geq 1 - \varepsilon, \qquad |\nabla v_i(x)| \leq \varepsilon, \qquad |A_x(\cdot, v_i)| + |(\nabla A)_x(\cdot, v_i)| \leq \varepsilon, \tag{6.3}$$

then $|\langle v_i(x), v_j(x)\rangle - \delta_{ij}| \leq 2\varepsilon$ (in fact this requires only the inequality on the left). Additionally, there exists $\varepsilon_1 = \varepsilon_1(n)$ such that if $\varepsilon \leq \varepsilon_1$, then

$$|\phi_0(x) - \phi(x)| + |\nabla^{\Sigma_0}(\phi_0(x) - \phi(x))| \leq C\varepsilon(1 + |\phi(x)| + |\nabla\phi(x)|),$$

where $C = C(n)$, and ϕ_0 is the ϕ of Σ_0.

Remark 6.15. In the original paper, this is stated for $k = n-1$ without tight constraints on ε, and is iterated $n-k$ times when applied. However, this overlooks the subtlety that ε must be small to allow repeated application of the lemma. We found it easier to instead generalise the lemma to arbitrary k, giving the statement above. Our proof generalises the $k = n-1$ case, but original arguments were needed to overcome high-dimensional complications.

Proof of Lemma 6.14. All computations in this proof are done at the point x, which will be suppressed in notation. Since $v_i = \partial_i - \langle\partial_i, \mathbf{n}\rangle\,\mathbf{n}$ where $\mathbf{n}$ is the unit normal to Σ at x, we have

$$\langle v_i, v_j\rangle = \langle\partial_i - \langle\partial_i, \mathbf{n}\rangle\,\mathbf{n}, \partial_j - \langle\partial_j, \mathbf{n}\rangle\,\mathbf{n}\rangle = \delta_{ij} - \langle\partial_i, \mathbf{n}\rangle\,\langle\partial_j, \mathbf{n}\rangle\,. \tag{6.4}$$

Since $|v_i|^2 = 1 - \langle\partial_i, \mathbf{n}\rangle^2$ and $|v_i| \geq 1 - \varepsilon$, we have

$$\langle\partial_i, \mathbf{n}\rangle \leq \sqrt{2\varepsilon}. \tag{6.5}$$

Combining this with (6.4) gives

$$|\langle v_i, v_j\rangle - \delta_{ij}| \leq 2\varepsilon, \tag{6.6}$$

which is the first claim. Next, apply Gram-Schmidt to the vectors $\{v_{k+2}, \dots, v_{n+1}\}$ to get an orthonormal set $\left\{\frac{w_{k+2}}{|w_{k+2}|}, \dots, \frac{w_{n+1}}{|w_{n+1}|}\right\}$, where each w_i is the formal determinant

$$w_i = \frac{1}{|v_{k+2}|^2 \dots |v_{i-1}|^2} \begin{vmatrix} \langle v_{k+2}, v_{k+2}\rangle & \langle v_{k+2}, v_{k+3}\rangle & \cdots & \langle v_{k+2}, v_{i-1}\rangle & v_{k+2} \\ \langle v_{k+3}, v_{k+2}\rangle & \langle v_{k+3}, v_{k+3}\rangle & \cdots & \langle v_{k+3}, v_{i-1}\rangle & v_{k+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle v_i, v_{k+2}\rangle & \langle v_i, v_{k+3}\rangle & \cdots & \langle v_i, v_{i-1}\rangle & v_i \end{vmatrix}. \tag{6.7}$$

First assume that $\varepsilon < \frac{1}{2}$. We may use (6.6) and (6.7) to get a constant $N = N(n)$ such that $\frac{1}{2} < 1 - N\varepsilon \le |w_i| \le 1$ for each $i \in \{k+2, \dots, n+1\}$. Namely,

- Expanding (6.7) and using the reverse triangle inequality together with $\frac{1}{2} < 1-\varepsilon \le |v_i| \le 1$, we get $|w_i| \ge 1 - \varepsilon + \mathcal{O}(\varepsilon)$, where the coefficients in $\mathcal{O}(\varepsilon)$ depend only on n. Thus, we can shrink ε to get $|w_i| \ge 1 - N\varepsilon$ for some $N = N(n)$. The bound $|w_i| \le 1$ follows trivially from the Gram-Schmidt algorithm. We may shrink ε further (depending on n) so that $N\varepsilon < \frac{1}{2}$.

We can also arrange that $|\nabla w_i| \le N\varepsilon$ (for a possibly different N); the idea is to apply ∇ to (6.7) and use $|\nabla v_i| \le \varepsilon$, the Cauchy–Schwarz inequality and $\frac{1}{2} < |v_i| < 1$. Likewise we can bound $|A(\cdot, w_i)| + |(\nabla A)(\cdot, w_i)| \le N\varepsilon$ using (6.3), the bilinearity of A and ∇A, and the aforementioned facts. If N was enlarged at any point, tighten ε again to ensure $N\varepsilon < \frac{1}{2}$. To summarise, choosing ε small enough ensures that for all $i \in \{k+2, \dots, n+1\}$,

$$1 > |w_i| \ge 1 - N\varepsilon > \frac{1}{2}, \qquad |\nabla w_i| \le N\varepsilon < \frac{1}{2}, \qquad |A(\cdot, w_i)| + |(\nabla A)(\cdot, w_i)| \le N\varepsilon < \frac{1}{2}. \tag{6.8}$$

If $e_1, \dots, e_k$ is an orthonormal frame for Σ_0 in a neighbourhood of x, then

$$e_1, \dots, e_k, \frac{w_{k+2}}{|w_{k+2}|}, \frac{w_{k+3}}{|w_{k+3}|}, \dots, \frac{w_{n+1}}{|w_{n+1}|}$$

is an orthonormal frame for Σ. Furthermore, if $\mathbf{n}_0 \in \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$ is the normal to Σ_0 at x, then

$$\mathbf{n}_0 = \frac{\mathbf{n} - \sum_{i=k+2}^{n+1} \langle \partial_i, \mathbf{n}\rangle \partial_i}{\alpha}, \qquad \alpha = \left|\mathbf{n} - \sum_{i=k+2}^{n+1} \langle \partial_i, \mathbf{n}\rangle \partial_i\right| = \left(1 - \sum_{i=k+2}^{n+1} \langle \partial_i, \mathbf{n}\rangle^2\right)^{1/2}. \tag{6.9}$$

Transversity ensures that $\alpha \neq 0$. Since $\overline{\nabla}_{e_i} e_j \in \mathbb{R}^{k+1}$, we have $\left\langle \overline{\nabla}_{e_i} e_j, \partial_\ell \right\rangle = 0$ for $\ell \in \{k+2, \dots, n+1\}$. Thus, (6.9) gives $\left\langle \overline{\nabla}_{e_i} e_j, \mathbf{n}_0 \right\rangle = \frac{1}{\alpha} \left\langle \overline{\nabla}_{e_i} e_j, \mathbf{n} \right\rangle$. It follows that

$$\begin{aligned} H - H_0 &= -\left(\sum_{i=1}^{k} A(e_i, e_i) + \sum_{j=k+2}^{n+1} A\left(\frac{w_j}{|w_j|}, \frac{w_j}{|w_j|}\right)\right) + \sum_{i=1}^{k} \left\langle \overline{\nabla}_{e_i} e_i, \mathbf{n}_0 \right\rangle \\ &= \frac{1-\alpha}{\alpha} \sum_{i=1}^{k} A(e_i, e_i) - \sum_{j=k+2}^{n+1} A\left(\frac{w_j}{|w_j|}, \frac{w_j}{|w_j|}\right) \\ &= \frac{\alpha - 1}{\alpha} H - \frac{1}{\alpha} \sum_{j=k+2}^{n+1} A\left(\frac{w_j}{|w_j|}, \frac{w_j}{|w_j|}\right). \end{aligned} \tag{6.10}$$

As $x \in \Sigma_0$, we have $x_{k+2} = \dots = x_{n+1} = 0$, so $\langle x_0, \mathbf{n}_0\rangle = \langle x, \mathbf{n}_0\rangle$ and $\langle x, \partial_i\rangle = 0$ for $i \in \{k+2, \dots, n+1\}$. Using this in (6.9), we get

$$\langle x, \mathbf{n}\rangle - \langle x_0, \mathbf{n}_0\rangle = \langle x, \mathbf{n}\rangle - \frac{1}{\alpha} \left\langle x, \mathbf{n} - \sum_{i=k+2}^{n+1} \langle \partial_i, \mathbf{n}\rangle \partial_i \right\rangle = \frac{\alpha - 1}{\alpha} \langle x, \mathbf{n}\rangle. \tag{6.11}$$

Combining (6.10) and (6.11) gives

$$\begin{aligned}\phi-\phi_0&=\frac{1}{2}(\langle x,\mathbf{n}\rangle-\langle x_0,\mathbf{n}_0\rangle)-(H-H_0)\\&=\frac{\alpha-1}{\alpha}\left(\frac{1}{2}\langle x,\mathbf{n}\rangle-H\right)+\frac{1}{\alpha}\sum_{j=k+2}^{n+1}A\left(\frac{w_j}{|w_j|},\frac{w_j}{|w_j|}\right)\\&=\frac{\alpha-1}{\alpha}\phi+\frac{1}{\alpha}\sum_{j=k+2}^{n+1}A\left(\frac{w_j}{|w_j|},\frac{w_j}{|w_j|}\right).\end{aligned}\tag{6.12}$$

To bound this, we may assume $\varepsilon\le\frac{1}{4n}$, so that by (6.9), (6.5) and $0<\alpha<1$, we have

$$\alpha>\alpha^2=1-\sum_{i=k+2}^{n+1}\langle\partial_i,\mathbf{n}\rangle^2\ge 1-2\varepsilon(n-k)\ge\frac{1}{2},$$

$$1-\alpha<1-\alpha^2=\sum_{i=k+2}^{n+1}\langle\partial_i,\mathbf{n}\rangle^2\le 2\varepsilon(n-k)\le 2n\varepsilon,$$

$$|\nabla\alpha|<2|\alpha||\nabla\alpha|=|\nabla\alpha^2|\le\sum_{i=k+2}^{n+1}|\nabla\langle\partial_i,\mathbf{n}\rangle^2|=\sum_{i=k+2}^{n+1}|\nabla|v_i|^2|\le 2\sum_{i=k+2}^{n+1}|\nabla|v_i||\le 2\sum_{i=k+2}^{n+1}|\nabla v_i|\le 2n\varepsilon.$$

The last line uses $|v_i|^2=1-\langle\partial_i,\mathbf{n}\rangle^2$ and the Kato inequality. Using these in (6.12) and keeping (6.8) in mind, it follows that

$$|\phi-\phi_0|\le 4n\varepsilon|\phi|+\frac{2}{|w_j|}\sum_{j=k+2}^{n+1}\left|A\left(\frac{w_j}{|w_j|},w_j\right)\right|\le 4n\varepsilon|\phi|+4nN\varepsilon.\tag{6.13}$$

Similarly, differentiating (6.12) gives

$$\begin{aligned}|\nabla(\phi-\phi_0)|&\le\left|\frac{\alpha-1}{\alpha}\right||\nabla\phi|+\frac{|\nabla\alpha|}{|\alpha|}|\phi|+\left|\frac{\alpha-1}{\alpha^2}\right||\nabla\alpha||\phi|+\frac{|\nabla\alpha|}{\alpha^2}\sum_{j=k+2}^{n+1}\frac{|A(w_j,w_j)|}{|w_j|^2}\\&\quad+\frac{1}{|\alpha|}\sum_{j=k+2}^{n+1}\left(\frac{|(\nabla A)(w_j,w_j)|}{|w_j|^2}+2\left|A\left(\nabla\frac{w_j}{|w_j|},\frac{w_j}{|w_j|}\right)\right|\right)\\&\le 4n\varepsilon|\nabla\phi|+4n\varepsilon|\phi|+16n^2\varepsilon^2|\phi|+16n^2N\varepsilon^2+4nN\varepsilon+16nN\varepsilon.\end{aligned}\tag{6.14}$$

Adding (6.13) and (6.14) gives the second claim of the lemma, since $\varepsilon^2<\varepsilon$ and $N=N(n)$. □

6.2.2 Proof of Proposition 6.10

Our proof is structured like the original proof in [CM15], and is in five steps. Steps 1 and 2 follow the original. In Step 3, we borrowed arguments from [Man14] and made sure to use our version of Lemma 6.14. Step 4 is different, and our version resolves what seems to be an error in the original proof (discussed in Remark 6.16). The constructions in Step 4 are similar to, but not the same as, those in [CM19b]. The original proof has no Step 5, but it seems necessary to complete the proof; we adapted ideas from [CIM15] in our explanation.

Proof of Proposition 6.10. All constants C will have dependence at most $C(n, K_1)$. We include illustrations for the case $n = 2$, $k = 1$.

Step 1: Fixing the model cylinder. By the $(\varepsilon_0, 5\sqrt{2n}, C^2)$-closeness to a cylinder in $\mathcal{C}_k$, at any point $p \in B_{2\sqrt{2n}} \cap \Sigma$ there are $n - k$ orthonormal eigenvectors

$$v_{k+2}(p), v_{k+3}(p), \ldots, v_{n+1}(p)$$

of A with eigenvalues $\kappa_{k+2}(p), \ldots, \kappa_{n+1}(p)$ of absolute value less than $\frac{1}{\sqrt{100n}}$, and k other eigenvalues greater than $\frac{1}{\sqrt{4n}}$. This assumes that ε_0 is chosen small depending on n. Since $H \geq \frac{1}{2}$ on $B_{2\sqrt{2n}} \cap \Sigma$ and $\|\nabla \tau\|_{C^1(B_{2\sqrt{2n}})} < 1$ (taking $\varepsilon_1 < 1$ say), Lemma 6.12 and $A = H\tau$ give that

$$|\kappa_i(p)| \leq C \|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})}, \quad k + 2 \leq i \leq n + 1. \tag{6.15}$$

Intuitively, the vectors $\{v_i(p)\}_{i=k+2}^{n+1}$ are the flat directions at p of the almost-cylinder $B_{5\sqrt{2n}} \cap \Sigma$. We will write Σ as a graph over (part of) the cylinder $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$, rotated to align the $\mathbb{R}^{n-k}$ directions with the flat directions. Without loss of generality, assume the flat directions coincide with the coordinate directions $x_{k+2}, \ldots, x_{n+1}$, so no rotation is required (see Figure 6.2). Extend the $v_i(p)$ to $n - k$ tangential vector fields v_i on Σ defined by

$$v_i(x) = \nabla x_i = v_i(p) - \langle v_i(p), \mathbf{n}(x) \rangle \, \mathbf{n}(x). \tag{6.16}$$

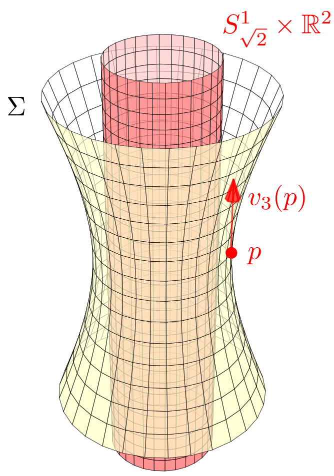

Figure 6.2: Using the vertical directions to Σ at p to determine the model cylinder.

Step 2: Bounds near p. For any $\rho \in [5\sqrt{2n}, r)$, let Ω_ρ be the set of points in $B_\rho \cap \Sigma$ which can be reached from p by a path in $B_\rho \cap \Sigma$ of length at most 3ρ. We will show that for all $x \in \Omega_\rho$,

$$|v_i(x) - v_i(p)| \leq C\rho^2 \|\nabla \tau\|_{C^1(B_\rho)}, \tag{6.17}$$

$$|\tau_x(v_i)| \leq C\rho^2 \|\nabla \tau\|_{C^1(B_\rho)}, \tag{6.18}$$

$$|\nabla_{v_i(x)} A| \leq C\rho^2 \|\nabla \tau\|_{C^1(B_\rho)}. \tag{6.19}$$

Since $\rho^2 \|\nabla \tau\|_{C^1(B_\rho)} \leq \varepsilon_1$ is small, these bounds convey that Ω_ρ is almost cylindrical.

Let $\gamma : [0, 3\rho] \to B_\rho \cap \Sigma$ be a curve with $\gamma(0) = p$ and $|\gamma'| \le 1$, and let w be a parallel unit vector field along γ with $w(0) = v_i(p)$. Then at any point $\gamma(s)$,

$$|\nabla_{\gamma'(s)}\tau(w)| \le C|\nabla\tau||\gamma'||w| \le C\,\|\nabla\tau\|_{C^1(B_\rho)}\,.$$

where $C = C(n)$ arises from the equivalence of norms (as we have conflated operator and tensor norms). Thus, integrating along γ gives

$$|\tau(w(s))| \le |\tau_p(v_i(p))| + 3\rho C\,\|\nabla\tau\|_{C^1(B_\rho)} \le C\rho\,\|\nabla\tau\|_{C^1(B_\rho)}\,, \tag{6.20}$$

where the last inequality uses (6.15). Since $|H| \le \sqrt{n}|A| \le K_1\sqrt{n}$, we then have

$$|A(w(s))| = |H||\tau(w(s))| \le C\rho\,\|\nabla\tau\|_{C^1(B_\rho)}\,.$$

Using this and $\overline{\nabla}_{\gamma'}w = A(\gamma', w)\mathbf{n}$, it holds for any $t \in [0, 3\rho]$ that

$$|w(t) - v_i(p)| = |w(t) - w(0)| \le \int_0^{3\rho} |A(w(s))|\,ds \le C\rho^2\,\|\nabla\tau\|_{C^1(B_\rho)}\,.$$

Since $w(t) \in T_{\gamma(t)}\Sigma$, and $v_i(\gamma(t))$ is the orthogonal projection of $v_i(p)$ onto $T_{\gamma(t)}\Sigma$, we have

$$|v_i(\gamma(t)) - v_i(p)| \le |w(t) - v_i(p)| \le C\rho^2\,\|\nabla\tau\|_{C^1(B_\rho)}\,,$$

giving (6.17). Likewise,

$$|w(t) - v_i(\gamma(t))| \le |w(t) - v_i(p)| \le C\rho^2\,\|\nabla\tau\|_{C^1(B_\rho)}\,,$$

so by (6.20) and the bound $|\tau| = C(n)|A|/H \le C(n, K_1)$ from (1) in the hypotheses, we get

$$|\tau_{\gamma(t)}(v_i)| \le |\tau(w(t))| + |\tau(w(t) - v_i(\gamma(t)))| \le C\rho^2\,\|\nabla\tau\|_{C^1(B_\rho)}\,,$$

which is (6.18). Finally, the Codazzi equations give that for any unit vector fields X and Y,

$$\begin{aligned}|(\nabla_{v_i}A)(X,Y)| = |(\nabla_x A)(v_i, Y)| &= |(\nabla_X(H\tau))(v_i, y)|\\ &= |H(\nabla_X\tau)(v_i, Y)| + |(\nabla_X H)\cdot\tau(v_i, Y)|\\ &\le C\,\|\nabla\tau\|_{C^1(B_\rho)} + C\rho^2\,\|\nabla\tau\|_{C^1(B_\rho)}\,,\end{aligned}$$

where the last inequality used (6.18) as well as the upper bounds on $|H|$ and $|\nabla H|$ induced by the inequality $|A| + |\nabla A| \le K_1$. This proves (6.19).

Step 3: Graphical bounds on a cross-section. By $(\varepsilon_0, 5\sqrt{2n}, C^2)$-closeness, the horizontal slice

$$\Sigma_0 = B_{5\sqrt{2n}} \cap \Sigma \cap \{x_{k+2} = \ldots = x_{n+1} = 0\}$$

is almost $S^k_{\sqrt{2k}} \subset \mathbb{R}^{k+1}$. We will apply Lemma 6.14 to bound ϕ_0 (the ϕ of Σ_0), then use Lemma 6.13 to get bounds when writing Σ_0 as the graph of a function u_0 over $S^k_{\sqrt{2k}}$ (see Figure 6.3).

First, we check that Σ_0 meets conditions (6.3) of Lemma 6.14. The estimates of Step 2 apply with $\rho = 5\sqrt{2n}$ since $\Sigma_0 \subset B_{5\sqrt{2n}} \cap \Sigma \subset \Omega_{5\sqrt{2n}}$ by the C^2-closeness. For each $x \in \Sigma_0$ and $i \in \{k+2, \ldots, n+1\}$,

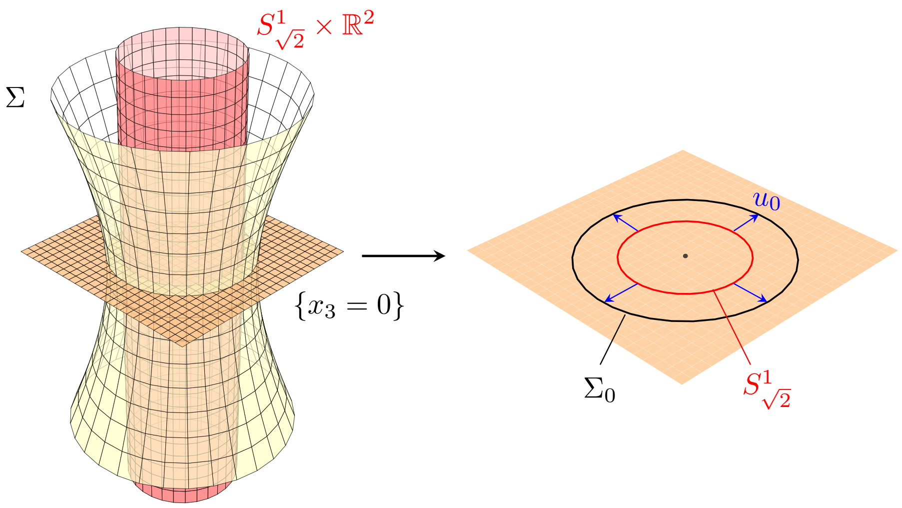

Figure 6.3: Slicing horizontally to get bounds on u_0.

(i) Using (6.17) and that $v_i(p)$ has unit length, we have

$$|v_i(x)| \geq |v_i(p)| - |v_i(x) - v_i(p)| \geq 1 - C \|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})},$$

where the $(5\sqrt{2n})^2$ factor from (6.17) was absorbed into C.

(ii) Where $\overline{\nabla}$ is the Euclidean derivative, (6.16) gives that for any vector field X on Σ,

$$\overline{\nabla}_X v_i(x) = -\langle v_i(p), \overline{\nabla}_X \mathbf{n}(x)\rangle\, \mathbf{n}(x) - \langle v_i(p), \mathbf{n}(x)\rangle\, \overline{\nabla}_X \mathbf{n}(x).$$

Projecting onto the tangent space of Σ and taking norms, we get

$$\begin{aligned} |\nabla_X v_i(x)| &= |(\langle v_i(x) - v_i(p), \mathbf{n}(x)\rangle - \langle v_i(x), \mathbf{n}(x)\rangle) A_x(X, \cdot)| \\ &\leq C(5\sqrt{2n})^2 \|\nabla \tau\|_{C^1(B_\rho)} |X| \leq C \|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})} |X|, \end{aligned}$$

where the first inequality uses (6.17) and the bound on $|A|$. Hence, $|\nabla v_i(x)| \leq C \|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})}$.

(iii) By (6.18), (6.19) and the upper bound on H,

$$|A_x(\cdot, v_i)| + |(\nabla A)_x(\cdot, v_i)| = |H||\tau_x(\cdot, v_i)| + |\nabla_{v_i(x)} A| \leq C \|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})}.$$

Taking ε_1 small depending on n and K_1, Lemma 6.14 applies with $\varepsilon = C \|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})}$ to give

$$\begin{aligned} \|\phi_0\|_{C^1(\Sigma_0)} &\leq C \|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})} (1 + \|\phi\|_{C^1(B_{5\sqrt{2n}})}) + \|\phi\|_{C^1(B_{5\sqrt{2n}})} \\ &\leq C(\|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})} + \|\phi\|_{C^1(B_{5\sqrt{2n}})}). \end{aligned}$$

Shrinking ε_0 further, Lemma 6.13 gives that Σ_0 is the graph over $S^k_{\sqrt{2k}}$ of a function u_0 with

$$\|u_0\|_{C^{2,\alpha}} \leq C(\|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})} + \|\phi\|_{C^1(B_{5\sqrt{2n}})}). \tag{6.21}$$

Step 4: Extending graphical bounds vertically within Ω_ρ. Let $w : \Sigma \to \mathbb{R}$ be the distance to the axis of the model cylinder $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$, so that

$$w(x) = \left(\sum_{j=1}^{k+1} x_j^2 \right)^{1/2},$$

and $w(x) = \sqrt{2k}$ if and only if $x \in S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$. If w_0 is w restricted to Σ_0, then (6.21) reads

$$\left\| w_0 - \sqrt{2k} \right\|_{C^{2,\alpha}(\Sigma_0)} \le C(\|\nabla\tau\|_{C^1(B_{5\sqrt{2n}})} + \|\phi\|_{C^1(B_{5\sqrt{2n}})}). \tag{6.22}$$

We will extend these graphical bounds vertically by subjecting Σ_0 to a family of flows. To construct these flows, define $(n-k)^2$ functions $J_{i\ell} = \langle v_i, v_\ell \rangle$, where $i, \ell \in \{k+2, \dots, n+1\}$. Within Ω_ρ, we have from (6.17) that

$$|v_i(x)| \ge |v_i(p)| - |v_i(x) - v_i(p)| \ge 1 - C\rho^2 \|\nabla\tau\|_{C^1(B_\rho)} \ge 1 - C\varepsilon_1,$$

so by taking ε_1 small, Lemma 6.14 gives $|J_{i\ell} - \delta_{i\ell}| \le 2C\varepsilon_1$. Then $(J_{i\ell})$ is close to the identity matrix, and has an inverse $(J^{i\ell})$ that is also close to the identity on Ω_ρ, say $|J^{i\ell} - \delta_{i\ell}| \le C$. For each $\xi = (\xi_{k+2}, \dots, \xi_{n+1}) \in \mathbb{R}^{n-k}$ with $|\xi| = 1$, define a vector field X^ξ on Σ by

$$X^\xi = \sum_{i,\ell=k+2}^{n+1} J^{i\ell} \xi_\ell v_i.$$

If γ is an integral curve of X^ξ, then for each $\beta \in \{k+2, \dots, n+1\}$,

$$\frac{d}{dt} x_\beta(\gamma(t)) = \left\langle \nabla x_\beta(\gamma(t)), X^\xi(\gamma(t)) \right\rangle = \left\langle v_\beta, J^{i\ell}\xi_\ell v_i \right\rangle = J^{i\ell} J_{\ell\beta} \xi_\ell = \xi_\beta.$$

Thus, flowing by X^ξ changes the 'height' $h(x) = (x_{k+2}, \dots, x_{n+1})$ at constant rate ξ. If we start the flow from Σ_0 (where we have $h = 0$), then at time t we get a connected component of $\Sigma \cap \{h = t\xi\}$ which is a topological S^k (see Figure 6.4).

Let $i \in \{k+2, \dots, n+1\}$ and $j \in \{1, \dots, k+1\}$. Since $\overline{\nabla} x_j = \nabla x_j + \langle \overline{\nabla} x_j, \mathbf{n} \rangle \mathbf{n}$ and $\overline{\nabla}^2 x_j = 0$, it holds on Ω_ρ that

$$|\overline{\nabla}_{v_i} \nabla x_j| = |\overline{\nabla}_{v_i}(\langle \overline{\nabla} x_j, \mathbf{n} \rangle \mathbf{n})| \le 2|\overline{\nabla}_{v_i}\mathbf{n}| = 2|A(v_i, \cdot)| \le 2|H||\tau(v_i)| \le C\rho^2 \|\nabla\tau\|_{C^1(B_\rho)}, \tag{6.23}$$

where the first inequality uses the Leibniz rule and Cauchy–Schwarz, and the last inequality uses (6.18) and $|H| \le \sqrt{n}|A| \le K_1\sqrt{n}$. Then since $X^\xi = J^{i\ell}\xi_\ell v_i$,

$$\sup_{\Omega_\rho} |\overline{\nabla}_{X^\xi} \nabla x_j| \le \sup_{\Omega_\rho} |J^{i\ell}||\xi_\ell||\overline{\nabla}_{v_i} \nabla x_j| \le C\rho^2 \|\nabla\tau\|_{C^1(B_\rho)}, \tag{6.24}$$

where the last inequality uses (6.23) and that $|J^{i\ell}| \le \delta_{i\ell} + C \le C$ and $|\xi_\ell| \le 1$. By (6.17), we have that on Ω_ρ,

$$|\nabla_{v_i} x_j| \le |\nabla_{v_i(p)} x_j| + |\nabla_{v_i(p) - v_i(x)} x_j| \le |\overline{\nabla}_{v_i(p)} x_j| + C\rho^2 \|\nabla\tau\|_{C^1(B_\rho)} |\nabla x_j| \le C\rho^2 \|\nabla\tau\|_{C^1(B_\rho)},$$

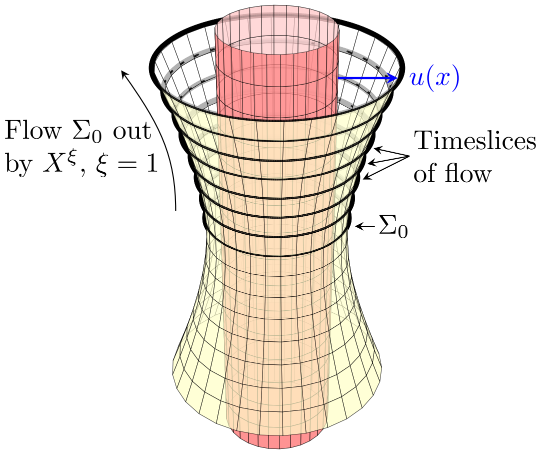

Figure 6.4: Flowing Σ_0 by X^ξ to extend the bounds on u_0 to bounds on u within Ω_ρ. Each timeslice of the flow is (part of) a level set of the 'height' function h.

where we used that $\langle \overline{\nabla} x_j, v_i(p) \rangle = 0$ and $|\nabla x_j| \leq |\overline{\nabla} x_j| = 1$. Reasoning as in (6.24), this gives $\sup_{\Omega_\rho} |\overline{\nabla}_{X^\xi} x_j| \leq C\rho^2 \|\nabla \tau\|_{C^1(B_\rho)}$. Finally, combining this and (6.24) gives

$$\sup_{\Omega_\rho} |\overline{\nabla}_{X^\xi} \nabla w^2| = 2 \sup_{\Omega_\rho} |\overline{\nabla}_{X^\xi} (x_j \nabla x_j)| \leq C\rho^2 \|\nabla \tau\|_{C^1(B_\rho)} \sup_{\Omega_\rho} (|\nabla x_j| + |x_j|) \leq C\rho^3 \|\nabla \tau\|_{C^1(B_\rho)} . \tag{6.25}$$

We now flow Σ_0 out by X^ξ to obtain the claimed pointwise bounds for $u = w - \sqrt{2k}$. Let $\Phi^\xi(q,t)$ be the flow of X^ξ from $q \in \Sigma_0$ for time t. We will flow for time $\sqrt{\rho^2 - 3k}$ and get bounds that hold for as long as the flow remains within Ω_ρ. That is, our bounds will hold on the set

$$\Omega_\rho^\xi = \{\Phi^\xi(q,t) \in \Sigma \mid q \in \Sigma_0, 0 \leq t \leq \sqrt{\rho^2 - 3k} \text{ and } \Phi^\xi(q,s) \in \Omega_\rho \text{ for all } s \leq t\}.$$

Integrating (6.25) up from Σ_0 and using (6.22), we get

$$\begin{aligned}
\sup_{\Omega_\rho^\xi} |\nabla w^2| &\leq \sup_{\Sigma_0} |\nabla w^2| + \rho \sup_{\Omega_\rho} |\overline{\nabla}_{X^\xi} \nabla w^2| \\
&\leq 2 \sup_{\Sigma_0} |w_0| \sup_{\Sigma_0} |\nabla w_0| + C\rho^4 \|\nabla \tau\|_{C^1(B_\rho)} \\
&\leq C(\|\nabla \tau\|_{C^1(B_{5\sqrt{2n}})} + \|\phi\|_{C^1(B_{5\sqrt{2n}})}) + C\rho^4 \|\nabla \tau\|_{C^1(B_\rho)} \\
&\leq C \|\phi\|_{C^1(B_{5\sqrt{2n}})} + C\rho^4 \|\nabla \tau\|_{C^1(B_\rho)} .
\end{aligned} \tag{6.26}$$

Integrating (6.26) up from Σ_0 and using (6.22), we get

$$\begin{aligned}
\sup_{\Omega_\rho^\xi} |w^2 - 2k| &\leq \sup_{\Sigma_0} |w_0^2 - 2k| + \rho \sup_{\Omega_\rho^\xi} |\nabla_{X^\xi} w^2| \\
&\leq \left(\sup_{\Sigma_0} |w_0 - \sqrt{2k}| \right) \left(\sup_{\Sigma_0} |w_0 - \sqrt{2k}| + 2\sqrt{2k} \right) + \rho \sup_{\Omega_\rho^\xi} (|X^\xi| |\nabla w^2|) \\
&\leq C\rho \|\phi\|_{C^1(B_{5\sqrt{2n}})} + C\rho^5 \|\nabla \tau\|_{C^1(B_\rho)} .
\end{aligned} \tag{6.27}$$

If we impose $C\rho \|\phi\|_{C^1(B_{5\sqrt{2n}})} + C\rho^5 \|\nabla \tau\|_{C^1(B_\rho)} \leq 1$ (via ε_1), then (6.27) yields $w \geq 1$ on Ω_ρ^ξ.

Thus, for the function $u = w - \sqrt{2k}$, (6.26) gives

$$\sup_{\Omega_\rho^\xi} |\nabla u| = \sup_{\Omega_\rho^\xi} |\nabla w| = \sup_{\Omega_\rho^\xi} \left| \frac{\nabla w^2}{2w} \right| \le C\rho \|\phi\|_{C^1(B_{5\sqrt{2n}})} + C\rho^4 \|\nabla \tau\|_{C^1(B_\rho)} . \tag{6.28}$$

Also, dividing (6.27) by $w + \sqrt{2k} > 1$ gives the same bound for $|u|$ over Ω_ρ^ξ. Adding this to (6.28) gives the C^1 bound for u claimed by the proposition, valid on Ω_ρ^ξ.

Step 5: Showing the bounds hold on all $B_r \cap \Sigma$. In this final step we will show that

$$B_\rho \cap \Sigma \subset \bigcup_{\xi \in \mathbb{R}^{n-k}, |\xi|=1} \Omega_\rho^\xi \tag{6.29}$$

for all $\rho \le r$, so the pointwise bounds on u from Step 4 hold on $B_r \cap \Sigma$, giving the proposition.

Take $\xi \in \mathbb{R}^{n-k}$ arbitrary with $|\xi| = 1$, and let

$$D = \{|h| = \sqrt{\rho^2 - 3k},\ 0 \le u \le \sqrt{3k}\} \subset B_\rho.$$

Since ε_1 is small, (6.17) gives that $(J^{i\ell})$ is almost the identity and v_i is almost equal to $v_i(p)$ on Ω_ρ. Thus, the flow vector field $X^\xi = J^{i\ell}\xi_\ell v_i \approx \xi_i v_i(p)$ is almost constant and directed vertically, as long as the flow remains within Ω_ρ.

Considering the C^2-closeness to a cylinder in $\mathcal{C}_k$ and that $p \in B_{2\sqrt{2n}}$, the path distance in Σ between p and a point on Σ_0 is at most $2\sqrt{2n} + \pi\sqrt{2k} + \sqrt{2n} < 10\sqrt{2n} < 2\rho$.[3] If we flow Σ_0 out by X^ξ for time $\sqrt{\rho^2 - 3k}$, we can tighten ε_1 to ensure that as long as the flow is within Ω_ρ,

- X^ξ is almost constant and vertical with unit magnitude. Thus the flow lines have length at most ρ, and $0 \le u \le \sqrt{3k}$ everywhere on the flow lines (recalling $u \approx \sqrt{2k}$ on Σ_0).

This implies the path distance in Σ from p to any point on a flow line is at most $2\rho + \rho = 3\rho$, and every flow line hits D before escaping Ω_ρ^ξ. As $D \subset B_\rho$, this shows that the connected component of $B_\rho \cap \Sigma$ containing Σ_0 is contained in $\bigcup_\xi \Omega_\rho^\xi$. But there is only one such component by hypothesis (4) of the proposition, so (6.29) follows and the proof is complete. □

Remark 6.16. In the original proof, Step 4 defines a single radial vector field instead of the X^ξ we used, but to our understanding this is ill-defined at Σ_0. We would need to extend the vector field to Σ_0 using bump functions and make sure the flow is well-defined starting from Σ_0. Our solution circumvents these difficulties. Our vector fields X^ξ are similar to those of [CM19b], but not exactly the same; ours are constructed so that we could still use some of the computations in the original Step 4 (with some adjustments).

6.3 The first Łojasiewicz inequality for cylinder-like hypersurfaces

In this section, we will combine the graphical proposition from the preceding section with Proposition 6.17 below to prove the first Łojasiewicz inequality, Theorem 6.7.

[3] $2\sqrt{2n}$ is from $p \in B_{2\sqrt{2n}}$; $\pi\sqrt{2k}$ is from the waist circumference of a cylinder in $\mathcal{C}_k$; $\sqrt{2n}$ is the tolerance.

6.3.1 Bounding $\nabla\tau$ beneath the cylindrical scale

The final ingredient we need is the next proposition, which bounds $\|\nabla\tau\|_{C^1}$ in terms of $\|\phi\|_{L^1}$ within a suitably chosen cylindrical scale. This is much needed because $\nabla\tau$ features in the graphical proposition but not in the Łojasiewicz inequality. Our statement uses stronger assumptions than the one in the paper; this is because the proof needs an $|\nabla^\ell\phi|$ bound from Lemma 6.6, but the original hypotheses do not seem to enable this.

Proposition 6.17 (1.28)**.** *Given n, λ_0, ℓ and K, there exists $\varepsilon_0 = \varepsilon_0(n)$ so that the following holds. If $\varepsilon \leq \varepsilon_0$ and $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface with $\lambda(\Sigma) \leq \lambda_0$, then for all $R \leq \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)$ and r such that $r + \frac{1}{1+r} < R - 1$, we have*

$$\|\nabla\tau\|_{C^1(B_r)} \leq CR^{2n}\left\{e^{-d_{\ell,n}\frac{(R-1)^2}{8}} + \|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}}\right\} e^{\frac{r^2}{8}},$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$, and $d_{\ell,n} \in (0,1) \nearrow 1$ as $\ell \to \infty$.

The proof of this uses two lemmas: a Gaussian L^2 bound on τ, and an interpolation inequality. We state these but exclude their proofs; the L^2 bound uses computations similar to §4.2 anyway, and the interpolation inequality would be too much of a digression for our liking.

Lemma 6.18 (1.25)**.** *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface. If $B_R \cap \Sigma$ is smooth with $H \geq \frac{1}{2}$ and $|A| \leq K$, then there exists $C = C(n, K)$ so that for $s \in (0, R)$ we have*

$$\int_{B_{R-s}} |\nabla\tau|^2 e^{-\frac{|x|^2}{4}} \leq C\left\{\frac{1}{s^2}\mathcal{H}^n(B_R \cap \Sigma)e^{-\frac{(R-s)^2}{4}} + \int_{B_R\cap\Sigma} \left(|\phi| + |\nabla^2\phi|\right) e^{-\frac{|x|^2}{4}}\right\}.$$

Lemma 6.19 (B.1)**.** *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface with $|A| + |\nabla^\ell A| \leq K$ on $B_{2r} \cap \Sigma$. For each $j \leq \ell$, there exists $C = C(n, \ell, j, K)$ so that if T is a smooth tensor on $B_{2r} \cap \Sigma$, then setting $a_{\ell,n,j} = \frac{\ell-j}{\ell+n}$, we have for the unweighted L^1 norms*

$$\left\|\nabla^j T\right\|_{L^\infty(B_r)} \leq C\left\{r^{-n-j}\,\|T\|_{L^1(B_r)} + \|T\|_{L^1(B_{2r})}^{a_{\ell,n,j}}\,\|\nabla^\ell T\|_{L^\infty(B_{2r})}^{1-a_{\ell,n,j}}\right\}.$$

Proof of Proposition 6.17. By Lemma 6.6, for $j \leq \ell$ we have $|\nabla^j A|, |\nabla^j\tau| \leq C(\varepsilon, j, K)$ within $B_R \cap \Sigma$. Let $y \in \Sigma$ have $|y| = r$, so that $|y| + \frac{1}{1+|y|} < R - 1$. Define the ball B^y and the constant δ_y by

$$B^y = B_{\frac{1}{1+|y|}}(y), \quad \delta_y = \int_{B^y\cap\Sigma} |\nabla\tau|,$$

so that $B^y \subset B_{R-1}$. Applying Lemma 6.19 to $\nabla\tau$ on the ball $\frac{1}{2}B^y = B_{\frac{1}{2(1+|y|)}}(y)$, we get

$$|\nabla\tau|(y) \leq C\left\{R^n\delta_y + \delta_y^{a_{\ell,n}}\|\nabla^\ell\tau\|_{L^\infty(B^y)}^{1-a_{\ell,n}}\right\} \leq C\left\{R^n\delta_y + \delta_y^{a_{\ell,n}}\right\} \leq CR^n\delta_y^{a_{\ell,n}}, \tag{6.30}$$

$$|\nabla^2\tau|(y) \leq C\left\{R^{n+1}\delta_y + \delta_y^{b_{\ell,n}}\|\nabla^\ell\tau\|_{L^\infty(B^y)}^{1-b_{\ell,n}}\right\} \leq C\left\{R^{n+1}\delta_y + \delta_y^{b_{\ell,n}}\right\} \leq CR^{n+1}\delta_y^{b_{\ell,n}}, \tag{6.31}$$

where $a_{\ell,n} = \frac{\ell}{\ell+n}$, $b_{\ell,n} = \frac{\ell-1}{\ell+n}$ and $C = C(n, \varepsilon, \ell, K)$. The first inequality in each line comes from $2(1 + |y|) < 2R$, while the second and third inequalities use the bounds on $|\nabla^\ell\tau|, |\nabla\tau|$.

We will bound $\delta_y^{a_{\ell,n}}$ and $\delta_y^{b_{\ell,n}}$ on the right of (6.30) and (6.31). For all $x \in B^y$, we have $|x|^2 \leq (|y| + \frac{1}{1+|y|})^2 \leq |y|^2 + 4$, where the second inequality is algebraic. Thus

$$\inf_{x\in B^y} e^{-\frac{|x|^2}{4}} \geq e^{-\frac{|y|^2}{4}-1}.$$

By Cauchy–Schwarz, the above inequality, polynomial volume growth,[4] and Lemma 6.18, we get

$$\begin{aligned}(1+|y|)^n e^{-\frac{|y|^2}{4}-1}\delta_y^2 &\le (1+|y|)^n e^{-\frac{|y|^2}{4}-1}\mathcal{H}^n(B^y\cap\Sigma)\int_{B^y\cap\Sigma}|\nabla\tau|^2\\ &\le C(1+|y|)^n\cdot\left(\frac{1}{1+|y|}\right)^n\int_{B_{R-1}\cap\Sigma}|\nabla\tau|^2 e^{-\frac{|x|^2}{4}}\\ &\le C\Big\{R^n e^{-\frac{(R-1)^2}{4}}+\int_{B_{R-\frac12}\cap\Sigma}(|\phi|+|\nabla^2\phi|)e^{-\frac{|x|^2}{4}}\Big\},\end{aligned}\tag{6.32}$$

where $C=C(n,\lambda_0,\varepsilon)$. To bound $|\nabla^2\phi|$ on the right, choose balls $B^i=B_{\frac{1}{1+|z_i|}}(z_i)$ such that

- The half-balls $\frac12 B^i=B_{\frac{1}{2(1+|z_i|)}}(z_i)$ collectively cover $B_{R-\frac12}\cap\Sigma$;
- Each $x\in B_{R-\frac12}\cap\Sigma$ is in at most $c=c(n)<\infty$ of the $\frac12 B^i$'s.

Set $r_i=\frac{1}{1+|z_i|}$ to simplify notation. Applying Lemma 6.19 on B^i and using Lemma 6.6 to bound $\|\nabla^\ell\phi\|_{L^\infty(B^i)}$, we get

$$\sup_{\frac12 B^i}|\nabla^2\phi|\le C\left\{r_i^{-n-2}\int_{B^i\cap\Sigma}|\phi|+\left(\int_{B^i\cap\Sigma}|\phi|\right)^{c_{\ell,n}}\right\},\tag{6.33}$$

where $C=C(n,\varepsilon,\ell,K)$ and $c_{\ell,n}=\frac{\ell-2}{\ell+n}$. Because $\frac12 e^{-\frac14\left(z-\frac{1}{1+z}\right)^2}\le e^{-\frac{z^2}{4}}\le 2e^{-\frac14\left(z+\frac{1}{1+z}\right)^2}$ for all $z\ge 0$, we have for each $x\in B^i$ that $\frac12 e^{-\frac{|z_i|^2}{4}}\le e^{-\frac{|x|^2}{4}}\le 2e^{-\frac{|z_i|^2}{4}}$. Together with (6.33), polynomial volume growth and Hölder's inequality for sums, this gives

$$\begin{aligned}\int_{B_{R-\frac12}\cap\Sigma}|\nabla^2\phi|e^{-\frac{|x|^2}{4}} &\le \sum_i\int_{\frac12 B^i\cap\Sigma}|\nabla^2\phi|e^{-\frac{|x|^2}{4}}\\ &\le C\sum_i\mathcal{H}^n\left(\frac12 B^i\cap\Sigma\right)\left[r_i^{-n-2}\int_{B^i\cap\Sigma}|\phi|+\left(\int_{B^i\cap\Sigma}|\phi|\right)^{c_{\ell,n}}\right]e^{-\frac{|z_i|^2}{4}}\\ &\le C\sum_i\left[r_i^{-2}\int_{B^i\cap\Sigma}|\phi|+r_i^n\left(\int_{B^i\cap\Sigma}|\phi|\right)^{c_{\ell,n}}\right]e^{-\frac{|z_i|^2}{4}}\\ &\le C\left\{R^2c\int_{B_R\cap\Sigma}|\phi|e^{-\frac{|x|^2}{4}}+\sum_i\left(\int_{B^i\cap\Sigma}|\phi|e^{-\frac{|x|^2}{4}}\right)^{c_{\ell,n}}\right\}\\ &\le C\left[R^2\|\phi\|_{L^1(B_R)}+\|\phi\|_{L^1(B_R)}^{c_{\ell,n}}\right],\end{aligned}\tag{6.34}$$

where $C=C(n,\lambda_0,\varepsilon,\ell,K)$. This dependence remains for the rest of the proof. By Lemma 6.6,

$$\|\phi\|_{L^1(B_R)}\le\int_{B_R\cap\Sigma}\sup_{B_R\cap\Sigma}|\phi|\,e^{-\frac{|x|^2}{4}}+\int_{\Sigma\setminus B_R}e^{-\frac{|x|^2}{4}}\le\max\left\{1,\sup_{B_R\cap\Sigma}|\phi|\right\}(4\pi)^{\frac n2}\lambda_0=C,\tag{6.35}$$

so $\|\phi\|_{L^1(B_R)}\le C\,\|\phi\|_{L^1(B_R)}^{c_{\ell,n}}$. Combining this with (6.32) and (6.34), it follows that

$$e^{-\frac{|y|^2}{4}-1}\delta_y^2\le(1+|y|)^n e^{-\frac{|y|^2}{4}-1}\delta_y^2\le CR^n e^{-\frac{(R-1)^2}{4}}+CR^2\,\|\phi\|_{L^1(B_R)}^{c_{\ell,n}}.$$

[4] Polynomial volume growth on Σ comes from $C^{2,\alpha}$ closeness to a cylinder, the relative volume element formula from Lemma B.1, and polynomial volume growth on cylinders in $\mathcal{C}_k$.

Rearranging for δ_y then using that $(a+b)^p \le 2^p(|a|^p+|b|^p)$ for all $p \ge 0$, we have

$$\delta_y \le \left\{ CR^n \left(e^{-\frac{(R-1)^2}{4}} + \|\phi\|_{L^1(B_R)}^{c_{\ell,n}} \right) e^{\frac{|y|^2}{4}} \right\}^{\frac{1}{2}} \le CR^{\frac{n}{2}} \left(e^{-\frac{(R-1)^2}{8}} + \|\phi\|_{L^1(B_R)}^{\frac{c_{\ell,n}}{2}} \right) e^{\frac{|y|^2}{8}}. \tag{6.36}$$

Substituting this into (6.30) and using $(a+b)^p \le 2^p(|a|^p+|b|^p)$ again gives

$$\begin{aligned} |\nabla \tau|(y) &\le CR^{\frac{3n}{2}} \left(e^{-\frac{(R-1)^2}{8}} + \|\phi\|_{L^1(B_R)}^{\frac{c_{\ell,n}}{2}} \right)^{a_{\ell,n}} e^{a_{\ell,n}\frac{|y|^2}{8}} \\ &\le CR^{\frac{3n}{2}} \left(e^{-a_{\ell,n}c_{\ell,n}\frac{(R-1)^2}{8}} + \|\phi\|_{L^1(B_R)}^{\frac{a_{\ell,n}c_{\ell,n}}{2}} \right) e^{\frac{|y|^2}{8}}, \end{aligned} \tag{6.37}$$

since $a_{\ell,n}, c_{\ell,n} < 1$. Likewise, using (6.36) in (6.31) yields

$$|\nabla^2 \tau|(y) \le CR^{\frac{3n+2}{2}} \left(e^{-b_{\ell,n}c_{\ell,n}\frac{(R-1)^2}{8}} + \|\phi\|_{L^1(B_R)}^{\frac{b_{\ell,n}c_{\ell,n}}{2}} \right) e^{\frac{|y|^2}{8}}. \tag{6.38}$$

Adding (6.37) and (6.38) gives the result with $d_{\ell,n} = \min\{a_{\ell,n}c_{\ell,n}, b_{\ell,n}c_{\ell,n}\}$. □

6.3.2 Proof of Theorem 6.7

We are ready to prove the first Łojasiewicz inequality for cylinder-like hypersurfaces.

Proof of Theorem 6.7. Let $r \in (5\sqrt{2n}, R)$. We first estimate the central quantity of Proposition 6.10, which is

$$r\,\|\phi\|_{C^1(B_{5\sqrt{2n}})} + r^5\,\|\nabla \tau\|_{C^1(B_r)}\,. \tag{6.39}$$

Let $a_{\ell,n} = \frac{\ell}{\ell+n}$, and take ℓ_0 large so that $a_{\ell,n} \ge \frac{3}{4}$. By Lemma 6.19 and Lemma 6.6, we bound

$$\begin{aligned} \|\phi\|_{L^\infty(B_{5\sqrt{2n}})} &\le C\Big\{\Big(\int_{B_{10\sqrt{2n}}\cap\Sigma} |\phi|\Big) + \Big(\int_{B_{10\sqrt{2n}}\cap\Sigma} |\phi|\Big)^{a_{\ell,n}} \|\nabla^\ell \phi\|_{L^\infty(B_{10\sqrt{2n}})}^{1-a_{\ell,n}}\Big\} \\ &\le C\left\{ \|\phi\|_{L^1(B_{10\sqrt{2n}})} + \|\phi\|_{L^1(B_{10\sqrt{2n}})}^{a_{\ell,n}} \right\} \le C\,\|\phi\|_{L^1(B_R)}^{3/4}, \end{aligned} \tag{6.40}$$

where $C = C(n, \varepsilon, \ell, K)$ and we assumed $R_0 \ge 10\sqrt{2n}$ in the last inequality. Similarly,

$$\|\nabla \phi\|_{L^\infty(B_{5\sqrt{2n}})} \le C\,\|\phi\|_{L^1(B_R)}^{3/4}. \tag{6.41}$$

Adding (6.40) and (6.41) gives

$$\|\phi\|_{C^1(B_{5\sqrt{2n}})} \le C\,\|\phi\|_{L^1(B_R)}^{3/4}. \tag{6.42}$$

Proposition 6.17 gives that if ε_0 is small enough (depending on n) and $r \le R-2$, then

$$\|\nabla \tau\|_{C^1(B_r)} \le CR^{2n} \left\{ e^{-d_{\ell,n}\frac{R^2}{8}} + \|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} \right\} e^{\frac{r^2}{8}}, \tag{6.43}$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$ and $\lim_{\ell\to\infty} d_{\ell,n} = 1$. Using (6.42) and (6.43), we can bound (6.39):

$$\begin{aligned} r\,\|\phi\|_{C^1(B_{5\sqrt{2n}})} + r^5\,\|\nabla\tau\|_{C^1(B_r)} &\le CR\,\|\phi\|_{L^1(B_R)}^{3/4} + CR^{2n+5}\left\{ e^{-d_{\ell,n}\frac{R^2}{8}} + \|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} \right\} e^{\frac{r^2}{8}} \\ &\le \check{C}R^{2n+5}\left\{ \|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} + e^{-d_{\ell,n}\frac{R^2}{8}} \right\} e^{\frac{r^2}{8}}, \end{aligned} \tag{6.44}$$

where $\check{C} = \check{C}(n, \lambda_0, \varepsilon, \ell, K)$, and the second inequality uses (6.35) to bound $\|\phi\|_{L^1(B_R)}^{3/4}$ by $\|\phi\|_{L^1(B_R)}^{d_{\ell,n}/2}$. To compactify notation, define $(\star_{R,r,\ell,n})$ and $\widetilde{C} = \widetilde{C}(n, \lambda_0, \varepsilon, \ell, K)$ by

$$(\star_{R,r,\ell,n}) = R^{2n+5}\left\{\|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} + e^{-d_{\ell,n}\frac{R^2}{8}}\right\} e^{\frac{r^2}{8}} \quad \text{and} \quad \widetilde{C} = \frac{\varepsilon_1}{\check{C}},$$

where $\varepsilon_1 = \varepsilon_1(n)$ is given by Proposition 6.10. We may suppose ε_0 is less than that of Proposition 6.10, and tight enough to guarantee $H \geq \frac{1}{2}$ on $B_R \cap \Sigma$ by Lemma 6.6. Thus, if r is such that $(\star_{R,r,\ell,n}) \leq \widetilde{C}$, then by (6.44),

$$r\,\|\phi\|_{C^1(B_{5\sqrt{2n}})} + r^5\,\|\nabla\tau\|_{C^1(B_r)} \leq \check{C}\widetilde{C} \leq \varepsilon_1. \tag{6.45}$$

Hence, by Proposition 6.10, $B_r \cap \Sigma$ is the graph of u over a cylinder $\Gamma \in \mathcal{C}_k$ with

$$\begin{aligned} |u(x)| + |\nabla u(x)| &\leq C\left\{r\,\|\phi\|_{C^1(B_{5\sqrt{2n}})} + r^5\,\|\nabla\tau\|_{C^1(B_{|x|})}\right\} \\ &\leq CR^{2n+5}\left\{\|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} + e^{-d_{\ell,n}\frac{R^2}{8}}\right\} e^{\frac{|x|^2}{8}}, \end{aligned} \tag{6.46}$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$ and we used (6.44). Note that (6.45) and (6.46) still hold for any smaller $\widetilde{C}$ as long as $(\star_{R,r,\ell,n}) \leq \widetilde{C}$. This is important as we will shrink $\widetilde{C}$ in the next theorem.

Let $r_* = \sup\{r \leq R-2 \mid (\star_{R,r,\ell,n}) \leq \widetilde{C}\}$. By (6.46), the fact that $u = w_\Gamma - \sqrt{2k}$, and $r_* < R$ and polynomial volume growth, we get

$$\begin{aligned} \int_{B_{r_*}\cap\Sigma} (w_\Gamma(x) - \sqrt{2k})^2 e^{-\frac{|x|^2}{4}} &\leq CR^{4n+10}\left\{\|\phi\|_{L^1(B_R)}^{d_{\ell,n}} + e^{-d_{\ell,n}\frac{R^2}{4}}\right\}\mathcal{H}^n\left(B_{r_*}\cap\Sigma\right) \\ &\leq CR^{5n+10}\left\{\|\phi\|_{L^1(B_R)}^{d_{\ell,n}} + e^{-d_{\ell,n}\frac{R^2}{4}}\right\}. \end{aligned} \tag{6.47}$$

Since $R \geq R_0$, we can demand that R_0 is large enough so that

$$R^{2n+5} e^{-d_{\ell,n}\frac{R^2}{8}} e^{\frac{(R-2)^2}{8}} \geq \widetilde{C}, \tag{6.48}$$

which in turn implies $(\star_{R,R-2,\ell,n}) \geq \widetilde{C}$. This makes R_0 have the dependencies stated in the theorem. By the definition of r_* and the fact that $(\star_{R,r,\ell,n})$ is increasing in the r slot, we must have $(\star_{R,r_*,\ell,n}) = \widetilde{C}$. Using this and the geometric inequality $|w_\Gamma(x) - \sqrt{2k}| \leq |x|$, we can bound

$$\begin{aligned} \int_{(B_R\setminus B_{r_*})\cap\Sigma} (w_\Gamma(x) - \sqrt{2k})^2 e^{-\frac{|x|^2}{4}} &\leq \mathcal{H}^n\left(B_R\cap\Sigma\right) R^2 e^{-\frac{r_*^2}{4}} \leq CR^{n+2}e^{-\frac{r_*^2}{4}} \\ &\leq \frac{C}{\widetilde{C}^2} R^{5n+12}\left\{\|\phi\|_{L^1(B_R)}^{d_{\ell,n}} + e^{-d_{\ell,n}\frac{R^2}{4}}\right\}. \end{aligned} \tag{6.49}$$

But $C/\widetilde{C}^2$ is just another C, so adding (6.47) and (6.49) completes the proof of Theorem 6.7. □

Remark 6.20. Had we not imposed the $R \geq R_0$ lower bound via (6.48), then we cannot say that $(\star_{R,r_*,\ell,n}) = \widetilde{C}$, and the last inequality in (6.49) fails. This is not addressed in the original paper.

Branching off the above proof yields a similar theorem which later supplies the improvement step (Theorem 6.27). Unlike the above proof where only the C^0 bound of Proposition 6.10 was used (in (6.47)), the proof below truly needs the C^1 bound for interpolation. In the theorem we will reuse the quantity $(\star_{R,r,\ell,n})$ from above, defined by

$$(\star_{R,r,\ell,n}) = R^{2n+5}\left\{\|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} + e^{-d_{\ell,n}\frac{R^2}{8}}\right\} e^{\frac{r^2}{8}}. \tag{6.50}$$

Theorem 6.21. *Let n be given, and let Theorem 6.7 provide $\varepsilon_0 = \varepsilon_0(n)$ and $\ell_0 = \ell_0(n)$. Suppose $\lambda_0 > 0$, $\varepsilon \leq \varepsilon_0$, $\ell \geq \ell_0$, $K > 0$, and $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface with $\lambda(\Sigma) \leq \lambda_0$. Then there exists R_0 so that whenever $R \in [R_0, \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)]$ and $\bar{\varepsilon} > 0$, we can find $\widetilde{C} = \widetilde{C}(n, \lambda_0, \varepsilon, \ell, K, \bar{\varepsilon})$ such that $B_{r_*} \cap \Sigma$ is graphical over a cylinder $\Gamma \in \mathcal{C}_k$ with $\|u\|_{C^1} \leq \bar{\varepsilon}$, where*

$$r_* = \sup\{r \leq R - 2 \mid (\star_{R,r_*,\ell,n}) \leq \widetilde{C}\}.$$

Moreover, it holds that $(\star_{R,r_*,\ell,n}) = \widetilde{C}$ and

$$\left\|w_\Gamma - \sqrt{2k}\right\|_{L^2(B_R\cap\Sigma)}^2 \leq CR^\rho \left\{\|\phi\|_{L^1(B_R)}^{d_{\ell,n}} + e^{-d_{\ell,n}\frac{R^2}{4}}\right\},$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$, $\rho = \rho(n)$ and $d_{\ell,n} \in (0, 1) \nearrow 1$ as $\ell \to \infty$.

Proof. Follow the proof of Theorem 6.7 up to and including (6.46). By then, we established that if $r \leq R - 2$ and $(\star_{R,r,\ell,n}) \leq \widetilde{C}$, then $B_r \cap \Sigma$ is the graph of u over a cylinder $\Gamma \in \mathcal{C}_k$ with

$$|u(x)| + |\nabla u(x)| \leq C(\star_{R,|x|,\ell,n}),$$

where C, $\widetilde{C}$ both depend on $n, \lambda_0, \varepsilon, \ell, K$. Use this with $r = r_*$, so that on $B_{r_*} \cap \Sigma$ we have

$$|u(x)| + |\nabla u(x)| \leq C(\star_{R,r_*,\ell,n}) \leq C\widetilde{C},$$

As stated after (6.46), this also holds if we shrink $\widetilde{C}$. Thus we assume $\widetilde{C} \leq \frac{\bar{\varepsilon}}{C}$ so the above reads

$$\|u\|_{C^1} \leq C\widetilde{C} \leq \bar{\varepsilon}.$$

Shrinking $\widetilde{C}$ this way introduces an $\bar{\varepsilon}$ dependence, which agrees with the theorem statement. To get the lower bound R_0, the equality $(\star_{R,r_*,\ell,n}) = \widetilde{C}$, and the claimed L^2 estimate for $w_\Gamma - \sqrt{2k}$, continue as in the proof of Theorem 6.7, starting from (6.47). □

6.4 The gradient Łojasiewicz inequality for cylinder-like hypersurfaces

In §5, the Łojasiewicz–Simon gradient inequality was established by Lyapunov–Schmidt reduction. This involved functional analysis over a compact manifold M, where M was eventually taken as the compact tangent flow in question. This made sense, since the timeslices of the RMCF were normal graphs of functions over M (at least for some times). However, when the tangent flow is cylindrical, the compact moving hypersurfaces are *never* graphs over the whole cylinder. We need new methods to obtain a gradient inequality suited to our purposes.

Let $\Gamma \in \mathcal{C}_k$. We will use the $\mathcal{F}_\Gamma$-functional, its Euler–Lagrange functional $\mathcal{M}_\Gamma^\nu$ and the linearisation L at zero. These were defined in (4.2), (4.7) and (4.8) respectively (replace Σ with Γ). Then L

is elliptic with finite-dimensional kernel $\mathcal{K}$. We let Π and $(\cdot)^\perp$ denote orthogonal projection to $\mathcal{K}$ using the Gaussian L^2 inner product (4.4).

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface. In this section, we will prove Theorem 6.8 as follows.

(1) In a region $B_R \cap \Sigma$ where Σ is the graph of u over a cylinder, we will estimate $|\mathcal{F}(B_R \cap \Sigma) - \mathcal{F}(\mathcal{C}_k)|$ when u is close to $\mathcal{K}$, and when u is far from $\mathcal{K}$.

(2) Outside B_R, we will use a cutoff which generates error terms. Together with the last step, this gives a preliminary estimate for $|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)|$, which is Theorem 6.22.

(3) We will use the first Łojasiewicz inequality to shape this estimate into the required form.

Step (1) is reminiscent of the proof of Theorem 5.3: there we divided the proof into estimating the difference in the energy functional $\mathcal{E}$ in a direction almost along $\mathcal{K}$, and the difference in a direction almost orthogonal to $\mathcal{K}$. Step (2) is new here, as Σ is not a global graph over the cylinder. Step (3) is also new, but not unexpected as Łojasiewicz originally used (5.1) to prove (5.2); we bypassed this earlier by taking (5.2) as given.

6.4.1 A preliminary gradient estimate

In this subsection, we prove a preliminary estimate for $|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)|$ which reads:

Theorem 6.22 (4.1). *Given n and λ_0, there exists $\bar{\varepsilon} = \bar{\varepsilon}(n)$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface with $\lambda(\Sigma) \leq \lambda_0$, and Σ is $(\bar{\varepsilon}, \widetilde{R}, C^2)$-close to a cylinder $\Gamma \in \mathcal{C}_k$ with graph function $\widetilde{u}$, then for any $\beta \in [0,1)$*

$$|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)| \leq C \left\{ \|\phi\|_{L^2(B_{\widetilde{R}} \cap \Sigma)}^{\frac{3+\beta}{2}} + (1 + \widetilde{R}^\rho) e^{-\frac{(3+\beta)(\widetilde{R}-1)^2}{16}} + \|\widetilde{u}\|_{L^2(B_{\widetilde{R}} \cap \Gamma)}^{\frac{3+\beta}{1+\beta}} \right\},$$

where $C = C(n, \lambda_0)$, $\rho = \rho(n)$.

To prove this, we need estimates that lead to a bound for $|\mathcal{F}(B_{\widetilde{R}} \cap \Sigma) - \mathcal{F}(\mathcal{C}_k)|$ in the two cases for $\widetilde{u}$ indicated in Step (1) above. This proceeds similarly to Lemma 5.4, but is more involved as one now has to deal with noncompactness of the cylinder. We will skip this rather tedious process, seeing as the original paper already covers it in detail and we found no issues with the presentation. The next two lemmas summarise the results of this analysis. Here we use a new quantity $\|\cdot\|_2$ for functions on a cylinder $\Gamma \in \mathcal{C}_k$, defined by

$$\|u\|_2 = \left\| u^2 + |\nabla u|^2 + |\nabla^2 u(\cdot, \mathbb{R}^{n-k})|^2 + (1 + |x|)^{-1} |\nabla^2 u| \right\|_{L^2},$$

where $\mathbb{R}^{n-k}$ refers to the flat directions of Γ.[5] We abbreviate $\mathcal{M}_\Gamma^\nu$ to $\mathcal{M}$ for the rest of this section.

Lemma 6.23 (3.11, 3.23). *Let $\Gamma \in \mathcal{C}_k$. There exists C_0 and $\mu > 0$ depending on n so that for all $u \in W^{2,2}(\Gamma)$,*

$$\|Lu\|_{L^2} \leq C_0 \|u\|_{W^{2,2}}, \tag{6.51}$$

$$\mu \|u^\perp\|_{W^{2,2}} \leq \|Lu\|_{L^2}, \tag{6.52}$$

$$\|u\|_2 \leq C_0 \|\Pi u\|_{L^2}^2, \tag{6.53}$$

$$\|u^\perp\|_2 \leq C_0 \|u\|_{C^2} \|u^\perp\|_{W^{2,2}}. \tag{6.54}$$

[5] This is not a norm, since $\|au\|_2 = a^2 \|u\|_2$ for $a \in \mathbb{R}$.

Lemma 6.24 (4.3). *Let $\Gamma \in \mathcal{C}_k$. There exist C_1 and ε depending on n so that if $u \in C^2(\Gamma)$ has $\|u\|_{C^2} \leq \varepsilon$, then*

$$\|\mathcal{M}u - Lu\|_{L^2} \leq C_1 \|u\|_2 ,$$
$$\left|\mathcal{F}_\Gamma(u) - \mathcal{F}(\mathcal{C}_k) - \frac{1}{2}\left\langle u^\perp, Lu^\perp \right\rangle_{L^2(\Gamma)}\right| \leq C_1 \|u\|_{L^2} \|u\|_2 .$$

Proof of Theorem 6.22. We first extend $\widetilde{u}$ to an entire graph over Γ so that Lemmas 6.23 and 6.24 can be used. Thus we fix a cutoff function ψ with $0 \leq \psi \leq 1$ which is one on $B_{\widetilde{R}-1}$ and zero outside $B_{\widetilde{R}}$. Define $u = \psi\widetilde{u}$ on $B_{\widetilde{R}}$, and $u = 0$ otherwise. Then

$$\|u\|_{C^2} \leq C \|\widetilde{u}\|_{C^2} \leq C_n \bar{\varepsilon}, \tag{6.55}$$

where C_n depends only on the C^2 norm of ψ, hence only on n. Since Σ and $\mathrm{graph}_\Gamma(u)$ coincide in $B_{\widetilde{R}-1}$ and $0 \leq \psi \leq 1$, Lemma 6.2 gives $C = C(n, \lambda_0)$ and $\rho = \rho(n)$ so that

$$|\mathcal{F}(\Sigma) - \mathcal{F}_\Gamma(u)| \leq (4\pi)^{-\frac{n}{2}} \int_{\Sigma \setminus B_{\widetilde{R}-1}} e^{-\frac{|x|^2}{4}} \leq C \widetilde{R}^\rho e^{-\frac{(\widetilde{R}-1)^2}{4}}. \tag{6.56}$$

Since $\mathcal{M}u \approx \phi$ on $B_{\widetilde{R}-1}$ (Proposition B.2, where ϕ is the ϕ of Σ), and $\mathcal{M}u = 0$ outside $B_{\widetilde{R}}$,

$$\|\mathcal{M}u\|_{L^2} \leq C \|\phi\|_{L^2(B_{\widetilde{R}-1} \cap \Sigma)} + \|\mathcal{M}u\|_{L^2(\Gamma \cap (B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}))} \leq C \|\phi\|_{L^2(B_{\widetilde{R}} \cap \Sigma)} + C_{n,\bar{\varepsilon}} e^{-\frac{(\widetilde{R}-1)^2}{8}}. \tag{6.57}$$

Here we used (6.55) to control $|\mathcal{M}u|$ by a constant depending on n and $\bar{\varepsilon}$ since $\mathcal{M}u$ depends on up to second derivatives of u (by Lemma B.1 and Proposition B.2), and that the volume of $\Gamma \cap (B_{\widetilde{R}} \setminus B_{\widetilde{R}-1})$ depends only on n. Now let

$$\mathcal{F}_0(u) = \mathcal{F}_\Gamma(u) - \mathcal{F}(\mathcal{C}_k).$$

If $\bar{\varepsilon}$ is small depending on n, then (6.55) bounds $\|u\|_{C^2}$ tightly enough so that Lemma 6.24 gives $C_1 = C_1(n)$ such that

$$|\,\|\mathcal{M}u\|_{L^2} - \|Lu\|_{L^2}\,| \leq C_1 \|u\|_2 , \tag{6.58}$$
$$\left|\mathcal{F}_0(u) - \frac{1}{2}\left\langle u^\perp, Lu^\perp \right\rangle_{L^2}\right| \leq C_1 \|u\|_{L^2} \|u\|_2 . \tag{6.59}$$

Let $\beta \in [0, 1)$. We will bound $|\mathcal{F}_0(u)|$ in two cases depending on how close u is to the kernel of L. We will then combine this with (6.56) to bound $|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)|$ as required.

Case 1: Suppose that u is almost normal to the kernel in the sense that

$$\|\Pi u\|_2 \leq \varepsilon \|u^\perp\|_{W^{2,2}}^{1+\beta}, \tag{6.60}$$

where $\varepsilon > 0$ is chosen below. Note that by (6.54), (6.60) and (6.55),

$$\|u\|_2 \leq 2\,\|\Pi u\|_2 + 2\|u^\perp\|_2 \leq 2(C_2\varepsilon + C_0 C_n \bar{\varepsilon})\|u^\perp\|_{W^{2,2}}, \tag{6.61}$$

where C_0 and C_n depend on n, and $\|u^\perp\|_{W^{2,2}}^{1+\beta} \leq C_2(n, \lambda_0, \bar{\varepsilon})\|u^\perp\|_{W^{2,2}}$. Now by (6.58), (6.52) and (6.61), there exists $\mu = \mu(n)$ so that

$$\begin{aligned}\|\mathcal{M}u\|_{L^2} &\geq \|Lu\|_{L^2} - C_1 \|u\|_2 \geq \mu\|u^\perp\|_{W^{2,2}} - C_1 \|u\|_2 \\ &\geq (\mu - 2C_1(C_2\varepsilon + C_0 C_n \bar{\varepsilon}))\|u^\perp\|_{W^{2,2}}.\end{aligned}$$

Shrink $\bar{\varepsilon}$ depending on n so that $C_0 C_1 C_n \bar{\varepsilon} < \frac{\mu}{8}$. Also choose $\varepsilon > 0$ depending on $n, \lambda_0, \bar{\varepsilon}$ so that $C_1 C_2 \varepsilon < \frac{\mu}{8}$. Then

$$\|\mathcal{M}u\|_{L^2} \geq \frac{\mu}{2}\|u^\perp\|_{W^{2,2}}. \tag{6.62}$$

We have tightened $\bar{\varepsilon}$ twice so far, both times depending on n. From now on, $\bar{\varepsilon} = \bar{\varepsilon}(n)$ will be fixed. All constants C in the rest of the proof depend on (at most) n and λ_0. By (6.59), (6.51), (6.61) and the triangle inequality $\|u\|_{L^2} \leq \|u^\perp\|_{L^2} + \|\Pi u\|_{L^2}$, we have

$$\begin{aligned} |\mathcal{F}_0(u)| &\leq C\|u^\perp\|_{L^2}\|u^\perp\|_{W^{2,2}} + C_1 \|u\|_{L^2} \|u\|_2 \\ &\leq C\|u^\perp\|_{L^2}\|u^\perp\|_{W^{2,2}} + C \|\Pi u\|_{L^2} \|u^\perp\|_{W^{2,2}}. \end{aligned} \tag{6.63}$$

Since $\|\Pi u\|_{L^2}^2 \leq C \left\|(\Pi u)^2\right\|_{L^2} \leq C \|\Pi u\|_2$ by Hölder's inequality and the entropy bound, we can bound the second term above:

$$C \|\Pi u\|_{L^2} \|u^\perp\|_{W^{2,2}} \leq C \|\Pi u\|_2^{1/2} \|u^\perp\|_{W^{2,2}} \leq C\|u^\perp\|_{W^{2,2}}^{\frac{3+\beta}{2}},$$

where the last inequality is (6.60). Using this with $\|u^\perp\|_{L^2} \leq \|u^\perp\|_{W^{2,2}}$, (6.63) becomes

$$|\mathcal{F}_0(u)| \leq C\|u^\perp\|_{W^{2,2}}^2 + C\|u^\perp\|_{W^{2,2}}^{\frac{3+\beta}{2}} \leq C\|u^\perp\|_{W^{2,2}}^{\frac{3+\beta}{2}}. \tag{6.64}$$

But now (6.62), (6.57) and Peter-Paul allow us to further estimate

$$|\mathcal{F}_0(u)| \leq C \|\mathcal{M}u\|^{\frac{3+\beta}{2}} \leq C \|\phi\|_{L^2(B_{\widetilde{R}})}^{\frac{3+\beta}{2}} + Ce^{-\frac{(3+\beta)(\widetilde{R}-1)^2}{16}}.$$

Case 2: Suppose that u is close to the kernel in that

$$\|\Pi u\|_2 > \varepsilon\|u^\perp\|_{W^{2,2}}^{1+\beta}, \tag{6.65}$$

where ε was chosen in Case 1. We also get a similar inequality for $\|u^\perp\|_2$ using (6.54):

$$\|u^\perp\|_2 \leq C_0 \|u\|_{C^2} \|u^\perp\|_{W^{2,2}} \leq C \|\Pi u\|_2^{\frac{1}{1+\beta}},$$

where we have used $\|u\|_{C^2} \leq \bar{\varepsilon} = \bar{\varepsilon}(n)$ and (6.65) (recall that ε depends on n, λ_0). This yields

$$\|u\|_2 \leq 2 \|\Pi u\|_2 + 2\|u^\perp\|_2 \leq C \|\Pi u\|_2^{\frac{1}{1+\beta}}. \tag{6.66}$$

We use (6.65) and (6.66) to continue off the first line of (6.63) (which does not depend on the Case 1 assumption), giving

$$|\mathcal{F}_0(u)| \leq 2C \|u\|_{L^2} \|\Pi u\|_2^{\frac{1}{1+\beta}}.$$

But (6.53) gives $\|\Pi u\|_2 \leq C_0 \|\Pi u\|_{L^2}^2 \leq C_0 \|u\|_{L^2}^2$, where $C_0 = C_0(n)$. Hence

$$|\mathcal{F}_0(u)| \leq 2C \|u\|_{L^2}^{\frac{3+\beta}{1+\beta}} \leq C \|\widetilde{u}\|_{L^2(B_{\widetilde{R}})}^{\frac{3+\beta}{1+\beta}}, \tag{6.67}$$

the last inequality due to the fact that u vanishes outside $B_{\widetilde{R}}$ and $u \leq \widetilde{u}$.

Adding (6.64) and (6.67) to account for both cases at once, we get

$$|\mathcal{F}_0(u)| \leq C \|\phi\|_{L^2(B_{\widetilde{R}} \cap \Sigma)}^{\frac{3+\beta}{2}} + Ce^{-\frac{(3+\beta)(\widetilde{R}-1)^2}{16}} + C \|\widetilde{u}\|_{L^2(B_{\widetilde{R}})}^{\frac{3+\beta}{1+\beta}}.$$

Finally, we use the cutoff bound (6.56) to get

$$\begin{aligned} |\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)| &\leq |\mathcal{F}(\Sigma) - \mathcal{F}_\Gamma(u)| + |\mathcal{F}_0(u)| \\ &\leq C \|\phi\|_{L^2(B_{\widetilde{R}} \cap \Sigma)}^{\frac{3+\beta}{2}} + C(1 + \widetilde{R}^\rho)e^{-\frac{(3+\beta)(\widetilde{R}-1)^2}{16}} + C \|\widetilde{u}\|_{L^2(B_{\widetilde{R}})}^{\frac{3+\beta}{1+\beta}}, \end{aligned}$$

which is the claimed estimate. □

6.4.2 Proof of Theorem 6.8

The preliminary estimate of Theorem 6.22 has an unwanted $\|\widetilde{u}\|_{L^2(B_{\widetilde{R}})}$ term. We will use the first Łojasiewicz to swallow this into the other terms, giving us the gradient Łojasiewicz inequality of Theorem 6.8. As we will be writing Σ as a graph over two possibly different cylinders, we need a way to make their graphical bounds compatible. The next lemma takes care of this.

Lemma 6.25 (4.50). *There exists $\varepsilon_0 = \varepsilon_0(n)$ so that if $\Gamma \in \mathcal{C}_k$, $5\sqrt{2n} \le R_1 < R_2$ and*

- *$B_{R_1} \cap \Sigma$ is graphical over Γ with the graph function u satisfying $\|u\|_{C^1} \le \varepsilon_0$;*
- *Σ is $(\varepsilon_0, R_2, C^{2,\alpha})$-close to a cylinder in $\mathcal{C}_k$,*

then for $\bar{R} = \min\{2R_1, R_2\}$, we have that $B_{\bar{R}} \cap \Sigma$ is graphical over Γ for an extension $\bar{u}$ of u satisfying $\|\bar{u}\|_{C^2} \le \bar{\varepsilon}$, where $\bar{\varepsilon} = \bar{\varepsilon}(n)$ is given by Theorem 6.22.

Proof of Theorem 6.8. Let ε_0 be the lesser of Theorem 6.7 and Lemma 6.25. Let $\bar{\varepsilon} = \bar{\varepsilon}(n)$ be given by Theorem 6.22. By Theorem 6.21, there exists $\widetilde{C} = \widetilde{C}(n, \lambda_0, \varepsilon, \ell, K)$ such that $B_{r_*} \cap \Sigma$ is graphical over some $\Gamma \in \mathcal{C}_k$ with $\|u\|_{C^1} \le \varepsilon_0$, where

$$r_* \le R - 2, \quad (\star_{R, r_*, \ell, n}) = \widetilde{C}.$$

By Lemma 6.25, for $\bar{R} = \min\{2r_*, R\}$ we can write $B_{\bar{R}} \cap \Sigma$ as a graph of $\bar{u}$ over Γ such that $\|\bar{u}\|_{C^2} \le \bar{\varepsilon}$. Moreover, Theorem 6.21 gives

$$\|\bar{u}\|^2_{L^2(B_{\bar{R}})} \le \|\bar{u}\|^2_{L^2(B_R)} \le C R^{\rho} \left\{ \|\phi\|^{d_{\ell,n}}_{L^1(B_R)} + e^{-d_{\ell,n} \frac{R^2}{4}} \right\}, \tag{6.68}$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$ and $\rho = \rho(n)$. Now Theorem 6.22 gives that for all $\beta \in [0, 1)$,

$$|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)| \le C' \left\{ \|\phi\|^{\frac{3+\beta}{2}}_{L^2(B_{\bar{R}})} + (1 + \bar{R}^{\rho'}) e^{-\frac{(3+\beta)(\bar{R}-1)^2}{16}} + \|\bar{u}\|^{\frac{3+\beta}{1+\beta}}_{L^2(B_{\bar{R}})} \right\}, \tag{6.69}$$

where $C' = C'(n, \lambda_0)$ and $\rho' = \rho'(n)$. We aim to express everything in terms of a power of R times $\|\phi\|_{L^2(B_R)}$, allowing for exponentially decaying terms in R. If $\bar{R} = R$, the first two terms are sorted, while the last term satisfies (by (6.68))

$$\|\bar{u}\|^{\frac{3+\beta}{1+\beta}}_{L^2(B_{\bar{R}})} \le C R^{\frac{\rho(3+\beta)}{2+2\beta}} \left\{ \|\phi\|^{d_{\ell,n} \frac{3+\beta}{2+2\beta}}_{L^1(B_R)} + e^{-\frac{d_{\ell,n}(3+\beta) R^2}{8(1+\beta)}} \right\}, \tag{6.70}$$

which is of the desired form. If $\bar{R} \ne R$, the second term of (6.69) also needs to be managed. In this case $\bar{R} = 2r_*$, so we use $(\star_{R, r_*, \ell, n}) = \widetilde{C}$ to estimate

$$\begin{aligned} e^{-\frac{\bar{R}^2}{8}} &= \left(e^{-\frac{r_*^2}{8}} \right)^4 = \widetilde{C}^{-4} R^{8n+20} \left\{ e^{-d_{\ell,n} \frac{R^2}{8}} + \|\phi\|^{\frac{d_{\ell,n}}{2}}_{L^1(B_R)} \right\}^4 \\ &\le C R^{8n+20} \left\{ e^{-d_{\ell,n} \frac{R^2}{2}} + \|\phi\|^{2d_{\ell,n}}_{L^2(B_R)} \right\}, \end{aligned} \tag{6.71}$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$, and the inequality uses $\|\phi\|_{L^1(B_R)} \le C(n, \lambda_0) \|\phi\|_{L^2(B_R)}$. We can assume $\bar{R} > 4$ so that $\left(\frac{\bar{R}-1}{\bar{R}}\right)^2 > \frac{1}{2}$. Raising (6.71) to the $\frac{3+\beta}{2}\left(\frac{\bar{R}-1}{\bar{R}}\right)^2 > \frac{3+\beta}{4}$ power, we get

$$\begin{aligned} e^{-\frac{(3+\beta)(\bar{R}-1)^2}{16}} &\leq \left(e^{-\frac{\bar{R}^2}{8}}\right)^{\frac{3+\beta}{4}} \leq CR^{(2n+5)(3+\beta)}\left\{e^{-d_{\ell,n}\frac{(3+\beta)R^2}{8}} + \|\phi\|_{L^2(B_R)}^{d_{\ell,n}\frac{3+\beta}{2}}\right\} \\ &\leq CR^{8n+20}\left\{e^{-\frac{(3+\beta)(R-1)^2}{16}} + \|\phi\|_{L^2(B_R)}^{d_{\ell,n}\frac{3+\beta}{2}}\right\}, \end{aligned} \tag{6.72}$$

where the last inequality assumes ℓ_0 is large so that $d_{\ell,n} \geq \frac{1}{2}$. Putting (6.70) and (6.72) back into (6.69), $|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)|$ is bounded above by $C(n,\lambda_0,\varepsilon,\ell,K)R^{\rho'}$ times

$$\|\phi\|_{L^2(B_{\bar{R}})}^{\frac{3+\beta}{2}} + \left(e^{-\frac{(3+\beta)(R-1)^2}{16}} + \|\phi\|_{L^2(B_R)}^{d_{\ell,n}\frac{3+\beta}{2}}\right) + \left(\|\phi\|_{L^2(B_R)}^{d_{\ell,n}\frac{3+\beta}{2+2\beta}} + e^{-\frac{d_{\ell,n}(3+\beta)R^2}{8(1+\beta)}}\right).$$

Finally, arguing as in (6.35), we can bound $\|\phi\|_{L^2(B_R)}$ independently of R. This allows to absorb the first and third terms into the fourth at the cost of another factor of $C = C(n,\lambda_0,\varepsilon,\ell,K)$. □

6.5 A scale comparison theorem

We now lay the groundwork for turning the gradient inequality, Theorem 6.8, into the discrete differential inequality (6.2) for an RMCF Σ_s. As explained in the synopsis, this is done by choosing β, ℓ and K large in Theorem 6.8. On the other hand, ε will not need to be tightened further. Hence, for the rest of this chapter, we reserve ε_0 to mean the $\varepsilon_0(n)$ of Theorem 6.8, and we will use $\varepsilon = \varepsilon_0$ in our cylindrical scales.

In this section, we will show how K should be chosen depending on β and ℓ (in fact just ℓ). We will choose K to bound both error terms in the gradient inequality by a power greater than one of $\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1})$. By the synopsis, this is a matter of choosing K large (given ℓ) so that

$$\mathbf{r}_{\varepsilon_0,\ell,K}(\Sigma_T) \geq (1+\mu)R(\Sigma_T) \tag{6.73}$$

for some $\mu > 0$. Recall that $R(\Sigma_T)$ is the shrinker scale of the RMCF at time T, defined by

$$e^{-\frac{R(\Sigma_T)^2}{2}} = \mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}) = \int_{T-1}^{T+1} \|\phi\|_{L^2(\Sigma_s)}^2.$$

The main theorem of this section says that if the RMCF is almost cylindrical for all times close to T, then to each ℓ there exists a K (here called C_ℓ) making (6.73) true.

Theorem 6.26 (Scale comparison, 5.3). *Given n and λ_0, there exist ε_E, R_1, $\{C_\ell\}_{\ell\in\mathbb{N}_0}$ and $\mu > 0$ such that the following holds. If Σ_s is an RMCF with $\lambda(\Sigma_s) \leq \lambda_0$, and there exists $R \in [R_1, R(\Sigma_T)]$ such that for all $s \in [T-\frac{1}{2}, T+1]$, Σ_s is $(\varepsilon_E, R, C^{2,\alpha})$-close to a cylinder (depending on s), then*

(i) $(1+\mu)R(\Sigma_T) \leq \mathbf{r}_{\varepsilon_0,\ell,C_\ell}(\Sigma_T)$ *for each $\ell \in \mathbb{N}_0$, and*

(ii) $\|\phi\|^2_{L^2(B_{(1+\mu)R(\Sigma_T)}\cap\Sigma_T)} \leq \overline{C}(\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}))$.

It is crucial that μ is independent of ℓ, because in the next section we will use this μ to select β and ℓ large. Once ℓ is chosen, this theorem automatically comes back to select $K = C_\ell$ to bound the error terms as mentioned above.

Theorem 6.26 is proved by an extension-improvement iteration scheme. The extension step turns $(\varepsilon_E, R, C^{2,\alpha})$-closeness into $(\varepsilon_0,(1+\mu)R, C^{2,\alpha})$-closeness as long as $R \leq R(\Sigma_T)$. The improvement step will turn this into $(\varepsilon_E,(1-\theta)(1+\mu)R, C^{2,\alpha})$-closeness, the point being that we are able to choose $\theta < \mu$ so that the $C^{2,\alpha}$ ε_E-closeness extends to a larger scale. Provided we are beneath the shrinker scale, the two steps can be iterated which eventually yields the theorem.

This section corresponds to Section 5 of the original paper [CM15]. However, the original version of Theorem 6.26 does not seem accurate to us, and some of the lemmas there are not used in the way that they are stated. While attempting to understand that part of the paper, we drew from Mantoulidis' notes [Man14] and Zhu's paper [Zhu20]. This section synthesises the statements and proofs from these sources, as well as our own arguments, into a careful proof of Theorem 6.26. Whilst our treatment is quite technical, we believe it is accurate.[6]

6.5.1 Improvement step

The improvement step will come from combining Theorem 6.21 with additional interpolation.

Theorem 6.27 (Improvement step)**.** *Given $\theta \in (0,1)$, there exists $\ell_0 = \ell_0(n,\theta)$ so that if $\varepsilon_E, \lambda_0, K, \overline{C}$ are positive and $\ell \geq \ell_0$, there exists R_2 such that the following holds. If Σ is a hypersurface with $\lambda(\Sigma) \leq \lambda_0$, $R_* \geq R_2$, and $R \in [R_2, R_*]$ is such that*

- *$R \leq \mathbf{r}_{\varepsilon_0,\ell,K}(\Sigma)$ and $\|\phi\|^2_{L^2(B_R\cap\Sigma)} \leq \overline{C}e^{-\frac{R_*^2}{2}}$,*

then Σ is $(\varepsilon_E,(1-\theta)R, C^{2,\alpha})$-close to a cylinder in $\mathcal{C}_k$.

Proof. Let $\bar\varepsilon > 0$, to be chosen shortly. We may take R_2 to be greater than the $R_0 = R_0(n,\lambda_0,\ell,K)$ of Theorem 6.21 (where we used $\varepsilon = \varepsilon_0(n)$). By Theorem 6.21, there exists $\widetilde{C} = \widetilde{C}(n,\lambda_0,\ell,K,\bar\varepsilon)$ such that $B_{r_*}\cap\Sigma$ is graphical over a cylinder in $\mathcal{C}_k$ with $\|u\|_{C^1} \leq \bar\varepsilon$, where

$$r_* = \sup\{r \leq R-2 \mid (\star_{R,r,\ell,n}) \leq \widetilde{C}\},$$

and $(\star_{R,r,\ell,n})$ is given by (6.50). We will select $\bar\varepsilon$ to get ε_E-tight $C^{2,\alpha}$ bounds by interpolation. Namely, for every $\eta > 0$ there exists $C_{\eta,\ell}$ such that

$$\|u\|_{C^{2,\alpha}} \leq \eta\,\|u\|_{C^{\ell+2}} + C_{\eta,\ell}\,\|u\|_{C^1} \leq \eta\,\|u\|_{C^{\ell+2}} + C_{\eta,\ell}\bar\varepsilon. \tag{6.74}$$

Since $r_* < R \leq \mathbf{r}_{\varepsilon_0,\ell,K}(\Sigma)$, we can control $\|u\|_{C^{\ell+2}}$ by $C(n,\ell,K)$. Thus, η can be selected depending on n, ℓ, K and ε_E so that $\eta\,\|u\|_{C^{\ell+2}} \leq \frac{\varepsilon_E}{2}$. Then choose $\bar\varepsilon$ small so that $C_{\eta,\ell}\bar\varepsilon \leq \frac{\varepsilon_E}{2}$. Substituting back into (6.74) gives

$$\|u\|_{C^{2,\alpha}} \leq \varepsilon_E. \tag{6.75}$$

Choosing $\bar\varepsilon$ this way means $\bar\varepsilon$ depends on $n, K, \varepsilon_E, \ell$. Then $\widetilde{C} = \widetilde{C}(n,\lambda_0,\ell,K,\bar\varepsilon) = \widetilde{C}(n,\lambda_0,\ell,K,\varepsilon_E)$.

The bound (6.75) is on $B_{r_*}\cap\Sigma$, but the theorem asserts that it holds on $B_{(1-\theta)R}\cap\Sigma$. It therefore remains to show $r_* \geq (1-\theta)R$. By the definition of r_*, this is equivalent to showing

$$(1-\theta)R \leq R-2 \quad \text{and} \quad (\star_{R,(1-\theta)R,\ell,n}) \leq \widetilde{C}. \tag{6.76}$$

[6] It would be too pedantic to list all of our contributions, as these mostly have to do with how we drew from these sources to reconcile the minute details in the proof of Theorem 6.26. As for our 'original' mathematical arguments, these are the proofs of Theorem 6.28 and Lemma 6.34 (though the results are by no means novel).

To get the inequality on the right, Hölder's inequality gives $C = C(\lambda_0, \overline{C})$ such that

$$\|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} \leq C \|\phi\|_{L^2(B_R)}^{\frac{d_{\ell,n}}{2}} \leq Ce^{-d_{\ell,n}\frac{R_*^2}{8}},$$

where $d_{\ell,n}$ is in the definition of $(\star_{R,r,\ell,n})$ and has $\lim_{\ell\to\infty} d_{\ell,n} = 1$. The second inequality is from the hypotheses. Choosing ℓ_0 large so that $d_{\ell,n} \geq 1 - \frac{\theta}{2}$, and keeping in mind $R \leq R_*$ and $(1-\theta)^2 < 1 - \theta$, we get

$$\begin{aligned}
(\star_{R,(1-\theta)R,\ell,n}) &= R^{2n+5}\left\{\|\phi\|_{L^1(B_R)}^{\frac{d_{\ell,n}}{2}} + e^{-d_{\ell,n}\frac{R^2}{8}}\right\} e^{\frac{(1-\theta)^2R^2}{8}} \\
&\leq CR^{2n+5}\left\{e^{-d_{\ell,n}\frac{R_*^2}{8}} + e^{-d_{\ell,n}\frac{R^2}{8}}\right\} e^{\frac{(1-\theta)R^2}{8}} \\
&\leq CR^{2n+5}e^{-d_{\ell,n}\frac{R^2}{8}} e^{\frac{(1-\theta)R^2}{8}} \\
&\leq CR^{2n+5}e^{-\frac{\theta R^2}{16}}.
\end{aligned}$$

Choose R_2 large to further bound this by $\widetilde{C}$, so the second inequality of (6.76) holds. In doing this, R_2 inherits the dependencies of C and $\widetilde{C}$, and also depends on θ. This is consistent with the theorem statement. Finally, increase R_2 depending on θ to get the first inequality in (6.76). □

6.5.2 Extension step

We turn to the extension step for this subsection.

Theorem 6.28 (Extension step). *Given n and λ_0, there exist $\varepsilon_E, R_1, \overline{C}, \{C_\ell\}_{\ell\in\mathbb{N}_0}$ and $\mu > 0$ such that the following holds. If Σ_s flows by RMCF and $\lambda(\Sigma_s) \leq \lambda_0$, and there exists $R \in [R_1, R(\Sigma_T)]$ such that for all $s \in [T - \frac{1}{2}, T+1]$, Σ_s is $(\varepsilon_E, R, C^{2,\alpha})$-close to a cylinder $\Gamma_s \in \mathcal{C}_k$ (depending on s), then for all $s \in [T - \frac{1}{2}, T+1]$:*

- *(i)* $(1+\mu)R \leq \mathbf{r}_{\varepsilon_0,\ell,C_\ell}(\Sigma_s)$ *for each* $\ell \in \mathbb{N}_0$, *and*
- *(ii)* $\|\phi\|^2_{L^2(B_{(1+\mu)R}\cap\Sigma_s)} \leq \overline{C}(\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}))$.

To prove this, we will:

- Use standard technical results for (R)MCF to uniformly bound all covariant derivatives of A in a multiplicatively larger ball $B_{(1+\mu)R}$, giving the constants C_ℓ. This is Proposition 6.32.
- Turn these curvature bounds into graphical bounds over a cylinder in the time interval $[T - \frac{1}{2}, T+1]$ using a uniform stability lemma for RMCF (Lemma 6.34). This together with the previous step yields conclusion (i) of the theorem.
- Obtain the $\|\phi\|^2_{L^2}$ bound of (ii) using a mean value inequality (Lemma 6.35).

We begin by stating three regularity results needed for Proposition 6.32. The first two are standard results due to White [Whi05] and Ecker–Huisken [EH91] respectively. Our statements are Theorem 5.6 and Proposition 3.22 in [Eck04] respectively.[7]

[7]The statements there are for MCF, but are rescaled to RMCF Σ_s by writing $M_t = \sqrt{-t}\Sigma_s$ where $t = -e^{s_0 - s}$ and M_t is the MCF with $M_{-1} = \Sigma_{s_0}$.

Lemma 6.29 (Brakke–White regularity theorem)**.** *Given n and λ_0, there exist ε and C_0 such that if Σ_s is an RMCF in $\mathbb{R}^{n+1}$ with $\lambda(\Sigma_s) \le \lambda_0$, and for some $\tau \in (0,1)$ and $x_0 \in \mathbb{R}^{n+1}$ we have*

$$(4\pi\tau)^{-\frac{n}{2}} \int_{\Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} \le 1+\varepsilon,$$

then for all $s \in \left[s_0 - \log\left(1-\frac{3\tau}{4}\right), s_0 - \log(1-\tau)\right)$ we have

$$\sup_{\Sigma_s \cap B_{\frac{\sqrt{\tau}}{2}}\left(e^{\frac{1}{2}(s-s_0)}x_0\right)} |A|^2 \le \frac{C_0}{\tau}.$$

Lemma 6.30 (Ecker–Huisken estimates)**.** *Suppose Σ_s is an RMCF in $\mathbb{R}^{n+1}$ on a time interval $(s_0 - \log(1-\tau+\rho^2), s_0 - \log(1-\tau))$, and there exists C_0 such that for all s in this interval, we have*

$$\sup_{\Sigma_s \cap B_\rho\left(e^{\frac{1}{2}(s-s_0)}x_0\right)} |A|^2 \le \frac{C_0}{\rho^2}.$$

Then for any $\theta \in (0,1)$ and $\ell \in \mathbb{N}$, it holds for all $s \in (s_0 - \log(1-\tau+\theta^2\rho^2), s_0 - \log(1-\tau))$ that

$$\sup_{\Sigma_s \cap B_{\theta\rho}\left(e^{\frac{1}{2}(s-s_0)}x_0\right)} |\nabla^\ell A|^2 \le \frac{C_\ell}{\rho^{2(\ell+1)}},$$

where $C_\ell = C_\ell(n,\ell,\theta,C_0)$.

The third regularity lemma bounds the change in the $\mathcal{F}_{x_0,\tau}$ functional on the RMCF over time.

Lemma 6.31 (5.15)**.** *Given $\varepsilon > 0$, $\tau \in (0,1)$, n and λ_0, there exists $R_0 = R_0(\varepsilon,\tau,n,\lambda_0)$ and $\sigma = \sigma(n,\varepsilon,\lambda_0) \ge 2$ so that if Σ_s flows by RMCF, $\lambda(\Sigma_s) \le \lambda_0$, $R \in [R_0, R(\Sigma_T)]$, $x_0 \in B_{R-\sigma}$ and*

$$(4\pi\tau)^{-\frac{n}{2}} \int_{\Sigma_{T+1}} e^{-\frac{|x-x_0|^2}{4\tau}} \le 1+\frac{\varepsilon}{2}, \tag{6.77}$$

then for all $s_0 \in [T-1, T+1]$ we have

$$(4\pi\tau)^{-\frac{n}{2}} \int_{\Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} \le 1+\varepsilon.$$

Proof. By Lemma 6.2, there exists $\sigma = \sigma(n,\varepsilon,\lambda_0)$ such that for all $y \in \mathbb{R}^{n+1}$ and all s,

$$(4\pi\tau)^{-\frac{n}{2}} \int_{\Sigma_s \setminus B_{\sigma\sqrt{\tau}}(y)} e^{-\frac{|x-y|^2}{4\tau}} \le \frac{\varepsilon}{4}. \tag{6.78}$$

Equation (5.21) in [CM15] reads[8]

$$\begin{aligned} \int_{B_R \cap \Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} \le& \int_{B_{R+2}\cap\Sigma_{T+1}} e^{-\frac{|x-x_0|^2}{4\tau}} + \int_{s_0}^{T+1}\int_{B_{R+2}\cap\Sigma_s} \phi^2 \\ &+ \left(1 + \frac{\frac{|x_0|}{\tau} + (\frac{1}{\tau}-1)(R+2)}{2}\right)\int_{s_0}^{T+1}\int_{B_{R+2}\cap\Sigma_s} |\phi| e^{-\frac{|x-x_0|^2}{4\tau}}. \end{aligned} \tag{6.79}$$

[8] We skip the derivation which is a rudimentary computation using Lemma 5.10 and equation (5.20) in [CM15].

Since $x_0 \in B_{R-\sigma} \subset B_{R-2}$ and $\tau \in (0,1)$, we have

$$1 + \frac{\frac{|x_0|}{\tau} + (\frac{1}{\tau} - 1)(R+2)}{2} \leq 1 + \frac{R+2}{2\tau} + \frac{R+2}{2\tau} - \frac{R+2}{2} \leq \frac{R+2}{\tau}.$$

Therefore, (6.79) becomes

$$\begin{aligned} \int_{B_R \cap \Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} \leq \int_{B_{R+2} \cap \Sigma_{T+1}} e^{-\frac{|x-x_0|^2}{4\tau}} + \int_{T-1}^{T+1} \int_{B_{R+2} \cap \Sigma_s} \phi^2 \\ + \frac{R+2}{\tau} \int_{T-1}^{T+1} \int_{B_{R+2} \cap \Sigma_s} |\phi| e^{-\frac{|x-x_0|^2}{4\tau}}. \end{aligned} \tag{6.80}$$

Using $\lambda(\Sigma_s) \leq \lambda_0$ and the Cauchy–Schwarz inequality, we can bound the third term:

$$\begin{aligned} \int_{T-1}^{T+1} \int_{B_{R+2} \cap \Sigma_s} |\phi| e^{-\frac{|x-x_0|^2}{4\tau}} &\leq \left(\int_{T-1}^{T+1} \int_{B_{R+2} \cap \Sigma_s} \phi^2 e^{-\frac{|x-x_0|^2}{4\tau}} \right)^{1/2} \left(\int_{T-1}^{T+1} \int_{B_{R+2} \cap \Sigma_s} e^{-\frac{|x-x_0|^2}{4\tau}} \right)^{1/2} \\ &\leq \sqrt{(4\pi\tau)^{\frac{n}{2}} 2\lambda_0} \left(\int_{T-1}^{T+1} \int_{B_{R+2} \cap \Sigma_s} \phi^2 \right)^{1/2} \\ &\leq e^{\frac{(R+2)^2}{8}} \sqrt{(4\pi\tau)^{\frac{n}{2}} 2\lambda_0} \left(\int_{T-1}^{T+1} \|\phi\|^2_{L^2(\Sigma_s)} \right)^{1/2} \\ &= e^{\frac{(R+2)^2}{8} - \frac{R(\Sigma_T)^2}{4}} \sqrt{(4\pi\tau)^{\frac{n}{2}} 2\lambda_0}, \end{aligned} \tag{6.81}$$

where the last equality is the definition of $R(\Sigma_T)$. For the second term in (6.80), we have

$$\int_{T-1}^{T+1} \int_{B_{R+2} \cap \Sigma_s} \phi^2 \leq e^{\frac{(R+2)^2}{4}} \int_{T-1}^{T+1} \|\phi\|^2_{L^2(\Sigma_s)} = e^{\frac{(R+2)^2}{4} - \frac{R(\Sigma_T)^2}{2}}. \tag{6.82}$$

Putting (6.81) and (6.82) back into (6.80), we compute

$$\begin{aligned} \int_{\Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} &= \int_{B_R \cap \Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} + \int_{\Sigma_{s_0} \setminus B_R} e^{-\frac{|x-x_0|^2}{4\tau}} \\ &\leq \int_{B_{R+2} \cap \Sigma_{T+1}} e^{-\frac{|x-x_0|^2}{4\tau}} + e^{\frac{(R+2)^2}{4} - \frac{R(\Sigma_T)^2}{2}} \\ &\quad + \frac{R+2}{\tau} e^{\frac{(R+2)^2}{8} - \frac{R(\Sigma_T)^2}{4}} \sqrt{(4\pi\tau)^{\frac{n}{2}} 2\lambda_0} + \int_{\Sigma_{s_0} \setminus B_{\sigma\sqrt{\tau}}(x_0)} e^{-\frac{|x-x_0|^2}{4\tau}} \\ &\leq (4\pi\tau)^{\frac{n}{2}} \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}\right) + e^{\frac{(R+2)^2}{4} - \frac{R(\Sigma_T)^2}{2}} + \frac{R+2}{\tau} e^{\frac{(R+2)^2}{8} - \frac{R(\Sigma_T)^2}{4}} \sqrt{(4\pi\tau)^{\frac{n}{2}} 2\lambda_0}, \end{aligned}$$

where the first inequality uses that $x_0 \in B_{R-\sigma}$ and so $B_{\sigma\sqrt{\tau}}(x_0) \subset B_R$, and the last inequality is (6.78) and (6.77). Since $R(\Sigma_T) \geq R \geq R_0$, we can select R_0 large so that the two terms on the right sum to less than $(4\pi\tau)^{\frac{n}{2}} \frac{\varepsilon}{4}$. This choice of R_0 depends on ε, τ, n and λ_0 as claimed. Then

$$\int_{\Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} \leq (4\pi\tau)^{\frac{n}{2}} \left(1 + \frac{3\varepsilon}{4}\right) + (4\pi\tau)^{\frac{n}{2}} \frac{\varepsilon}{4} = (4\pi\tau)^{\frac{n}{2}} \varepsilon.$$

Multiplying both sides by $(4\pi\tau)^{-\frac{n}{2}}$ gives the result. □

Using the previous three lemmas, we will now assemble our first building block in the proof of the extension step.

Proposition 6.32 (5.6). *Given n and λ_0, there exists $\sigma \geq 2$ and $\delta > 0$ such that the following holds. Given $\tau \in (0,1)$, there exists $R_0 = R_0(n, \lambda_0, \tau)$ so that if Σ_s flows by the RMCF, $\lambda(\Sigma_s) \leq \lambda_0$, $R \in [R_0, R(\Sigma_T)]$, $x_0 \in B_{R-\sigma}$ and*

$$\sup_{\Sigma_{T+1} \cap B_{\sigma\sqrt{\tau}}(x_0)} |A|^2 \leq \frac{\delta}{\tau},$$

then for each $\ell \in \mathbb{N}_0$, there is a constant $C_\ell = C_\ell(n, \ell)$ such that the curvature bound

$$\sup_{\Sigma_s \cap B_{\frac{\sqrt{\tau}}{3}}\left(\frac{1}{\sqrt{1-\tau}} x_0\right)} |\nabla^\ell A|^2 \leq \frac{C_\ell}{\tau^{\ell+1}}$$

holds for all $s \in [T - 1 - \log(1-\tau), T + 1 - \log(1-\tau)]$.

Proof. Let Lemma 6.29 provide $\varepsilon = \varepsilon(n, \lambda_0)$, and let Lemma 6.31 provide $R_0 = R_0(\varepsilon, n, \lambda_0, \tau) = R_0(n, \lambda_0, \tau)$ and $\sigma = \sigma(n, \varepsilon, \lambda_0) = \sigma(n, \lambda_0)$. Select $\delta = \delta(\varepsilon) = \delta(n, \lambda_0)$ such that

$$\sup_{B_{\sigma\sqrt{\tau}}(x_0) \cap \Sigma_{T+1}} |A|^2 \leq \frac{\delta}{\tau} \quad \text{implies} \quad (4\pi\tau)^{-\frac{n}{2}} \int_{\Sigma_{T+1}} e^{-\frac{|x-x_0|^2}{4\tau}} \leq 1 + \frac{\varepsilon}{2}. \tag{6.83}$$

This is possible since small curvature near x_0 implies almost-flatness, making the integral on the right almost one. The left-hand side of (6.83) is satisfied by the hypotheses, so the right-hand side holds; now Lemma 6.31 gives that if $s_0 \in [T-1, T+1]$, then

$$(4\pi\tau)^{-\frac{n}{2}} \int_{\Sigma_{s_0}} e^{-\frac{|x-x_0|^2}{4\tau}} \leq 1 + \varepsilon.$$

By Lemma 6.29, it holds for each $s \in \left[s_0 - \log\left(1 - \frac{3\tau}{4}\right), s_0 - \log(1-\tau)\right]$ that

$$\sup_{\Sigma_s \cap B_{\frac{\sqrt{\tau}}{2}}\left(e^{\frac{1}{2}(s-s_0)} x_0\right)} |A|^2 \leq \frac{C_0}{\tau},$$

where $C_0 = C_0(n, \lambda_0)$. Using this bound, we may now apply Lemma 6.30 with $\rho = \frac{\sqrt{\tau}}{2}$ and $\theta = \frac{2}{3}$ to get that for all $s \in \left[s_0 - \log\left(1 - \frac{8\tau}{9}\right), s_0 - \log(1-\tau)\right]$ and $\ell \in \mathbb{N}_0$,

$$\sup_{\Sigma_s \cap B_{\frac{\sqrt{\tau}}{3}}\left(e^{\frac{1}{2}(s-s_0)} x_0\right)} |\nabla^\ell A|^2 \leq \frac{C_\ell}{\tau^{\ell+1}}, \tag{6.84}$$

where $C_\ell = C_\ell(n, \ell, C_0) = C_\ell(n, \ell, \lambda_0)$. In particular, by selecting $s = s_0 - \log(1-\tau)$ for each $s_0 \in [T-1, T+1]$, we allow s to take every value in the interval $[T-1-\log(1-\tau), T+1-\log(1-\tau)]$. By (6.84) we get for each such s that

$$\sup_{\Sigma_s \cap B_{\frac{\sqrt{\tau}}{3}}\left(e^{-\frac{1}{2}\log(1-\tau)} x_0\right)} |\nabla^\ell A|^2 = \sup_{\Sigma_s \cap B_{\frac{\sqrt{\tau}}{3}}\left(e^{\frac{1}{2}(s-s_0)} x_0\right)} |\nabla^\ell A|^2 \leq \frac{C_\ell}{\tau^{\ell+1}}.$$

Since $e^{-\frac{1}{2}\log(1-\tau)} = \frac{1}{\sqrt{1-\tau}}$, this proves the proposition. □

The second element in the proof of the extension step is a uniform stability lemma. For MCF, this is stated as follows:

Lemma 6.33 (5.39)**.** *Given n, ε, K and ξ, there exist $\delta, \theta > 0$ such that if $R > \sqrt{2n}$ and M_t is an MCF with*

(1) M_{-1} is $(\delta, R+2, C^{2,\alpha})$-close to a cylinder $\Gamma \in \mathcal{C}_k$;

(2) $\|A\|_{C^3(B_{R+2}\cap M_t)} \leq K$ for $t \in [-1-\xi, -1+\xi]$,

then for each $t \in [-1-\theta, -1+\theta]$, M_t is $(\varepsilon, R, C^{2,\alpha})$-close to $\sqrt{-t}\,\Gamma$.

See [CM15] for the proof. What we need is a variant of this lemma for RMCF. Since this is not a direct rescaling of Lemma 6.33 and we could not locate a proof, we will give one here.

Lemma 6.34. *Given n, K_0, ζ, there exists $\eta > 0$ such that if $R_2 > \sqrt{2n}$ and Σ_s is a RMCF with*

(1) Σ_{s_0} is $(\eta, R_2+2, C^{2,\alpha})$-close to a cylinder $\Gamma \in \mathcal{C}_k$;

(2) $\|A\|_{C^3(B_{R_2+2}\cap \Sigma_s)} \leq K_0$ for $s \in [s_0-\zeta, s_0+\zeta]$,

then for sufficiently small $\rho > 0$, there exists $\mu = \mu(\rho)$ so that $\Sigma_{s_0+\rho}$ is $(\varepsilon_0, (1+\mu)R_2, C^{2,\alpha})$-close to Γ.

Proof. The flow $M_t = \sqrt{-t}\Sigma_s$, $t = -e^{s_0-s}$ defines an MCF with $M_{-1} = \Sigma_{s_0}$. By (1), M_{-1} is $(\eta, R_2+2, C^{2,\alpha})$-close to Γ. Moreover, rescaling (2) gives $\xi = \xi(\zeta)$ such that $\|A\|_{C^3(B_{R_2+2}\cap M_t)} \leq 2K_0$ for $t \in [-1-\xi, -1+\xi]$ (the factor of 2 enters because the curvatures will be rescaled too in a temporal neighbourhood of $t = -1$). Feeding into Lemma 6.33 the parameters $R = R_2$, $\varepsilon = \varepsilon_0/2$, $K = 2K_0$ and ξ, we get $\delta, \theta > 0$ such that if M_{-1} is $(\delta, R_2+2, C^{2,\alpha})$-close to Γ, then the conclusion of that lemma holds with the supplied parameters. So let $\eta = \delta$, and we have that

- For each $t \in [-1-\theta, -1+\theta]$, M_t is $\left(\frac{\varepsilon_0}{2}, R_2, C^{2,\alpha}\right)$-close to $\sqrt{-t}\,\Gamma$.

Since $M_t = \sqrt{-t}\Sigma_s$, rescaling everything by a spatial factor of $\sqrt{-t}$ gives us that

- For each $t \in [-1-\theta, -1+\theta]$, $\Sigma_s = \Sigma_{s(t)}$ is $\left(\frac{\varepsilon_0}{2\sqrt{-t}}, \frac{R_2}{\sqrt{-t}}, C^{2,\alpha}\right)$-close to Γ.

If $t \in \left(-1, -\frac{1}{4}\right)$, then $\frac{1}{\sqrt{-t}} = 1+\mu < 2$ for some $\mu = \mu(t) = \mu(s) > 0$. Thus the above statement gives that in particular,

- For each $t \in \left(-1, \min\{-1+\theta, -\frac{1}{4}\}\right)$, $\Sigma_{s(t)}$ is $\left(\varepsilon_0, (1+\mu)R_2, C^{2,\alpha}\right)$-close to Γ.

Since s is increasing in t and $s(-1) = s_0$, we can replace $t \in \left(-1, \min\{-1+\theta, -\frac{1}{4}\}\right)$ in the above statement with $s \in (s_0, s_0+\overline{\rho})$, where $\overline{\rho} > 0$ is the value of s when $t = \min\{-1+\theta, -\frac{1}{4}\}$. This proves the lemma. □

The third ingredient for the extension step is a mean value inequality for the $\mathcal{F}$-functional, whose proof is a plain computation and we omit. We only remark that a bound on $|A|$ suffices rather than bounds on higher derivatives (as the original paper suggests). This is evident from the proof, specifically equation (5.37) in the paper.

Lemma 6.35 (5.32)**.** *Given n and C_1, there is a constant C so that if Σ_s is an RMCF for $s \in [s_1, s_2]$,*

$\beta \in (0, \frac{s_2-s_1}{2})$, and $|A| \leq C_1$ on $B_{r+1} \cap \Sigma$ for all $s \in [s_1, s_2]$, then

$$\max_{s\in[s_1+\beta,s_2]} \|\phi\|^2_{L^2(B_r\cap\Sigma_s)} \leq (C + 1/\beta)(\mathcal{F}(\Sigma_{s_1}) - \mathcal{F}(\Sigma_{s_2})).$$

We will now prove Theorem 6.28 using the three ingredients on hand. To execute this properly, we must carefully play off the constants provided in each part against one another and ensure no circular dependencies arise.

Proof of Theorem 6.28. Let $\sigma \geq 2$ and $\delta > 0$ be given by Proposition 6.32, depending on n and λ_0. As δ can be taken small, we assume $\delta \leq \frac{1}{2}(1 - e^{-1/4})$ for later convenience. Observe that if $x_0 \in B_{R-\sigma}$ and $\tau \in (0, 1)$, then $B_{\sigma\sqrt{\tau}}(x_0) \subset B_R$. Then since Σ_{T+1} is $(\varepsilon_E, R, C^{2,\alpha})$-close to $\Gamma_{T+1} \in \mathcal{C}_k$, we can find $\tau \in (0, 1)$ so that for all $x_0 \in B_{R-\sigma}$,

$$\sup_{\Sigma_{T+1}\cap B_{\sigma\sqrt{\tau}}(x_0)} |A|^2 \leq \frac{\delta}{\tau}.$$

As the left-hand quantity is controlled using only ε_E (and perhaps n, λ_0), this means τ is independent of R and x_0. Moreover, since any cylinder in $\mathcal{C}_k$ has $|A|^2 = \frac{1}{2}$, we must have $\tau \leq 2\delta$. We can also shrink ε_E (depending on n) so that the left-hand quantity is bounded by 1, in turn implying $\tau \geq \delta$. By Proposition 6.32, there exist $\{\widetilde{C}_\ell\}_{\ell\in\mathbb{N}_0}$ depending on n and ℓ so that

$$\sup_{\Sigma_s\cap B_{\frac{\sqrt{\tau}}{3}}\left(\frac{1}{\sqrt{1-\tau}}x_0\right)} |\nabla^\ell A|^2 \leq \frac{\widetilde{C}_\ell}{\tau^{\ell+1}} \tag{6.85}$$

for all $s \in [T - 1 - \log(1 - \tau), T + 1 - \log(1 - \tau)]$ and $x_0 \in B_{R-\sigma}$. The fact that (6.85) holds for *all* $x_0 \in B_{R-\sigma}$ means that these curvature bounds hold on a ball $B_{\frac{R-\sigma}{\sqrt{1-\tau}}} \cap \Sigma_s$ for each s. As $R \geq R_1$, setting R_1 sufficiently large (depending on τ and σ) will give that $\frac{R-\sigma}{\sqrt{1-\tau}} > (1 + \mu)R$ for some $\mu > 0$, so the curvature bounds hold on a multiplicatively larger ball. The fact that τ does not depend on R is important here. In summary, (6.85) gives

$$\sup_{\Sigma_s\cap B_{(1+\mu)R}} |\nabla^\ell A| \leq \left(\frac{\widetilde{C}_\ell}{\tau^{\ell+1}}\right)^{1/2} = C_\ell, \tag{6.86}$$

for all $s \in [T-1-\log(1-\tau), T+1-\log(1-\tau)]$, where the last equality *defines* C_ℓ. As $\delta \leq \tau \leq 2\delta \leq 1 - e^{-1/4}$, this time interval includes $[T - \frac{3}{4}, T + 1 - \log(1 - \delta)]$, hence $[T - \frac{1}{2}, T + 1]$ in particular. Therefore, to prove claim (i) of the theorem, it remains to show that Σ_s is $(\varepsilon_0, (1+\mu)R, C^{2,\alpha})$-close to a cylinder in $\mathcal{C}_k$ for each $s \in [T - \frac{1}{2}, T + 1]$. We will do this now.

By (6.86), we have $\|A\|_{C^3(B_R\cap\Sigma_s)} \leq K_0 = C_0 + C_1 + C_2 + C_3$ for all $s \in [T - \frac{3}{4}, T + 1 - \log(1 - \delta)]$. Let $s_0 \in [T - \frac{5}{8}, T + 1]$ and apply Lemma 6.34 centred at Σ_{s_0} with parameters $R_2 = R - 2$, K_0, and $\zeta = -\log(1 - \delta) < \frac{1}{8}$. The η given by that lemma depends on n, K_0 and ζ, which in our case all depend on n and λ_0. Thus, as long as $\varepsilon_E < \eta$, the lemma gives $\rho < \frac{1}{8}$ and $\mu > 0$ (both independent of s_0) such that $\Sigma_{s_0+\rho}$ is $(\varepsilon_0, (1 + \mu)(R - 2), C^{2,\alpha})$-close to Γ_{s_0}. Trading for a smaller μ, we get $(\varepsilon_0, (1 + \mu)R, C^{2,\alpha})$-closeness of $\Sigma_{s_0+\rho}$ to Γ_{s_0}. But $s_0 \in [T - \frac{5}{8}, T + 1]$ was arbitrary and $\rho < \frac{1}{8}$, so $s_0 + \rho$ can take any value between $T - \frac{1}{2}$ and $T + 1$. The claim (i) follows.

For (ii), shrink μ slightly so that the bounds (6.86) actually hold on $B_{(1+\mu)R+1} \cap \Sigma_s$. Invoking Lemma 6.35 with $s_1 = T - \frac{3}{4}$, $s_2 = T+1$, $r = (1+\mu)R$ and $\beta = \frac{1}{4}$, we get $\overline{C} > 0$ such that

$$\|\phi\|^2_{L^2(B_{(1+\mu)R}\cap\Sigma_s)} \le \overline{C}(\mathcal{F}(\Sigma_{T-\frac{3}{4}}) - \mathcal{F}(\Sigma_{T+1})) \le \overline{C}(\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}))$$

for all $s \in [T - \frac{1}{2}, T+1]$, where the last inequality is Lemma 3.24. This proves (ii). □

6.5.3 Proof of Theorem 6.26

We are prepared to use the extension and improvement steps from the last two subsections to prove the scale comparison theorem. For ease of viewing, let us state the precise versions of the extension and improvement steps that will be used.

Theorem 6.36 (Extension step). *Given n and λ_0, there exist $\varepsilon_E, R_1, \overline{C}, \{C_\ell\}_{\ell\in\mathbb{N}_0}$ and $\mu > 0$ such that the following holds. If Σ_s flows by RMCF and $\lambda(\Sigma_s) \le \lambda_0$, and there exists $R \in [R_1, R(\Sigma_T)]$ such that for all $s \in [T - \frac{1}{2}, T+1]$:*

$$(I_R): \quad \Sigma_s \text{ is } (\varepsilon_E, R, C^{2,\alpha})\text{-close to a cylinder in } \mathcal{C}_k \text{ (depending on } s),$$

then for all $s \in [T - \frac{1}{2}, T+1]$:

$$(E_{(1+\mu)R}): \quad \begin{cases} (1+\mu)R \le \mathbf{r}_{\varepsilon_0,\ell,C_\ell}(\Sigma_s) \text{ for each } \ell \in \mathbb{N}_0, \text{ and} \\ \|\phi\|^2_{L^2(B_{(1+\mu)R}\cap\Sigma_s)} \le \overline{C}(\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1})). \end{cases}$$

The version of the improvement step we will use follows from applying Theorem 6.27 to each timeslice of an RMCF, using $R_* = R(\Sigma_T)$, and replacing K with a sequence of numbers $\{C_\ell\}_{\ell\in\mathbb{N}_0}$.

Theorem 6.37 (Improvement step). *Given n, λ_0, $\varepsilon_E, \overline{C}$, $\{C_\ell\}_{\ell\in\mathbb{N}_0}$ and $\theta \in (0,1)$, there exists R_2 such that the following holds. If Σ_s flows by RMCF and $\lambda(\Sigma_s) \le \lambda_0$, and there exists $R \in [R_2, R(\Sigma_T)]$ such that for all $s \in [T - \frac{1}{2}, T+1]$:*

$$(E_R): \quad \begin{cases} R \le \mathbf{r}_{\varepsilon_0,\ell,C_\ell}(\Sigma_s) \text{ for each } \ell \in \mathbb{N}_0, \text{ and} \\ \|\phi\|^2_{L^2(B_R\cap\Sigma_s)} \le \overline{C}(\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1})), \end{cases}$$

then for all $s \in [T - \frac{1}{2}, T+1]$:

$$(I_{(1-\theta)R}): \quad \Sigma_s \text{ is } (\varepsilon_E, (1-\theta)R, C^{2,\alpha})\text{-close to a cylinder in } \mathcal{C}_k.$$

Remark 6.38. With these abbreviations, Theorem 6.26 essentially says that

$$(I_R) \text{ for all } s \in [T - \frac{1}{2}, T+1] \qquad \text{implies} \qquad (E_{(1+\mu)R(\Sigma_T)}) \text{ for } s = T.$$

Proof of Theorem 6.26. Let $\varepsilon_E, R_1, \mu, K$ and $\{C_\ell\}_{\ell\in\mathbb{N}_0}$ be provided by the extension step, depending on n, λ_0. By the extension step, $(E_{(1+\mu)R})$ holds for each $s \in [T - \frac{1}{2}, T+1]$. Feeding in $n, \lambda_0, \varepsilon_E, \overline{C}, \{C_\ell\}_{\ell\in\mathbb{N}_0}$ and $\theta = \mu/2$ into the improvement step, we obtain R_2 such that if $(1+\mu)R \ge R_2$, then $(I_{(1-\theta)(1+\mu)R})$ holds for each $s \in [T - \frac{1}{2}, T+1]$. Since R_2 does not depend on R_1, we may take R_1 large to make sure this holds.

Applying the extension step at scale $(1-\theta)(1+\mu)R > R$, we get that $(E_{(1+\mu)^2(1-\theta)R})$ holds for each $s \in [T - \frac{1}{2}, T+1]$. We may continue alternating between extension and improvement as long as we are beneath the shrinker scale $R(\Sigma_T)$, so the final iteration of the improvement step gives that $(I_{R(\Sigma_T)})$ holds for each $s \in [T-\frac{1}{2}, T+1]$. *The point is that the constants $\varepsilon_E, \overline{C}, \{C_\ell\}_{\ell\in\mathbb{N}_0}, \mu, R_1, \theta, R_2$ do not change between iterations, because their dependencies all trace back to n and λ_0 which are fixed.* Nonetheless, at scale $R(\Sigma_T)$ we apply the extension step one last time to get $(E_{(1+\mu)R(\Sigma_T)})$ for each $s \in [T-\frac{1}{2}, T+1]$. Choosing $s = T$ gives the theorem. □

6.6 Proof of Theorem 6.1

The proof of uniqueness of cylindrical tangent flows is now in sight. To begin this home stretch, we take the long-awaited step of turning the gradient Łojasiewicz inequality into the discrete differential inequality (6.2).

Theorem 6.39 (6.1). *Given n and λ_0, there exist $R_0, R_1, \varepsilon_E, C$ and $\delta \in (0,1)$ so that if Σ_s is an RMCF with $\lambda(\Sigma_s) \leq \lambda_0$, $R(\Sigma_T) \geq R_0$, and Σ_s is $(\varepsilon_E, R_1, C^{2,\alpha})$-close to a cylinder for each $s \in [T-\frac{1}{2}, T+1]$, then*

$$(\mathcal{F}(\Sigma_T) - \mathcal{F}(\mathcal{C}_k)) \leq C(\mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1}))^{\frac{1+\delta}{2}}.$$

Proof. Let R_1 and ε_E be those of Theorem 6.26. Then we get positive constants $\mu, \overline{C}$ and $\{C_\ell\}_{\ell\in\mathbb{N}_0}$ depending on n and λ_0 such that

- $(1+\mu)R(\Sigma_T) \leq \mathbf{r}_{\varepsilon_0,\ell,C_\ell}(\Sigma_T)$ for each $\ell \in \mathbb{N}_0$, and
- $\|\phi\|^2_{L^2(B_{(1+\mu)R(\Sigma_T)}\cap\Sigma_T)} \leq \overline{C}\mathcal{F}_T$,

where we have written $\mathcal{F}_T = \mathcal{F}(\Sigma_{T-1}) - \mathcal{F}(\Sigma_{T+1})$ to simplify notation.

Select $\tilde{\mu} < \mu$ so that $(1+\mu)R(\Sigma_T) > (1+\tilde{\mu})R(\Sigma_T) + 1$. Apply Theorem 6.8 to Σ_T at scale $R = (1+\tilde{\mu})R(\Sigma_T) + 1 \leq \mathbf{r}_{\varepsilon_0,\ell,C_\ell}(\Sigma_T)$. This gives that for any $\beta \in [0,1)$ and $\ell \geq \ell_0$,

$$|\mathcal{F}(\Sigma_T) - \mathcal{F}(\mathcal{C}_k)| \leq CR^\rho \left\{ \|\phi\|_{L^2(B_R\cap\Sigma_T)}^{d_{\ell,n}\frac{3+\beta}{2+2\beta}} + e^{-d_{\ell,n}\left(\frac{3+\beta}{2+2\beta}\right)\frac{R^2}{4}} + e^{-\frac{(3+\beta)(R-1)^2}{16}} \right\}, \tag{6.87}$$

where $C = C(n,\lambda_0,\ell,C_\ell) = C(n,\lambda_0,\ell)$, $\rho = \rho(n)$, and $d_{\ell,n} \in (0,1)$ has $\lim_{\ell\to\infty} d_{\ell,n} = 1$. This holds as long as $R \geq R_0(n,\lambda_0,\ell)$, but we know this is true since $R(\Sigma_T) \geq R_0$.

We will choose β and ℓ large to bound each term in (6.87) by a power greater than $\frac{1}{2}$ of $\mathcal{F}_T$. Once we do this, the C and R_0 above will depend only on n and λ_0 as claimed. For the last term in (6.87), note that $(R-1)^2 \geq (1+\tilde{\mu})R(\Sigma_T)^2$, so

$$e^{-\frac{(3+\beta)(R-1)^2}{16}} \leq e^{-\frac{(3+\beta)(1+\tilde{\mu})}{8}\frac{R(\Sigma_T)^2}{2}} = \mathcal{F}_T^{\frac{3+\beta}{4}\frac{1+\tilde{\mu}}{2}},$$

where the equality is the definition of $R(\Sigma_T)$. Now choose $\beta' \in [0,1)$ to make the exponent on the right equal to $\frac{1+\delta}{2}$ for some $\delta > 0$. To mitigate the R^ρ prefactor, choose $\beta > \beta'$ so that

$$R^\rho e^{-\frac{(3+\beta)(R-1)^2}{16}} = R^\rho \mathcal{F}_T^{\frac{1+\delta}{2}} e^{-\frac{(\beta-\beta')(R-1)^2}{16}} \leq C\mathcal{F}_T^{\frac{1+\delta}{2}}, \tag{6.88}$$

for some $C = C(\rho) = C(n)$. Next, choose ℓ' large so that

$$d_{\ell',n}\left(\frac{3+\beta}{2+2\beta}\right) \geq 1+\delta,$$

possibly choosing a smaller δ. For the second term in (6.87), since $R > R(\Sigma_T)$, we have

$$e^{-d_{\ell',n}\left(\frac{3+\beta}{2+2\beta}\right)\frac{R^2}{4}} \leq e^{-\frac{1+\delta}{2}\frac{R(\Sigma_T)^2}{2}} = \mathcal{F}_T^{\frac{1+\delta}{2}}. \tag{6.89}$$

Finally, for the first term, since $R < (1+\mu)R(\Sigma_T)$ we have

$$\|\phi\|_{L^2(B_R)}^{d_{\ell',n}\frac{3+\beta}{2+2\beta}} \leq \{\overline{C}\mathcal{F}_T\}^{\frac{1}{2}d_{\ell',n}\left(\frac{3+\beta}{2+2\beta}\right)} \leq C\mathcal{F}_T^{\frac{1+\delta}{2}}, \tag{6.90}$$

where $C = C(n, \lambda_0)$. Similarly to (6.88), we can pick $\ell > \ell'$ to absorb the R^ρ prefactor as $d_{\ell,n} > d_{\ell',n}$. Thus, (6.89) and (6.90) become

$$R^\rho e^{-d_{\ell,n}\left(\frac{3+\beta}{2+2\beta}\right)\frac{R^2}{4}} \leq C\mathcal{F}_T^{\frac{1+\delta}{2}},$$
$$R^\rho \|\phi\|_{L^2(B_R)}^{d_{\ell,n}\frac{3+\beta}{2+2\beta}} \leq C\mathcal{F}_T^{\frac{1+\delta}{2}}.$$

Combining these with (6.88), the right-hand side of (6.87) is bounded above by $C\mathcal{F}_T^{\frac{1+\delta}{2}}$. □

The following lemma captures the importance of the above theorem: it forces a certain finiteness to the 'length' of an RMCF. We will skip the proof, which is elementary but inevitably cumbersome.

Lemma 6.40 (6.9). *If $f : [0,\infty) \to [0,\infty)$ is a nonincreasing function, and there exist $\theta \in (\frac{1}{2}, 1)$ and $C > 0$ so that for all sufficiently large t we have*

$$f(t) \leq C(f(t-1) - f(t+1))^\theta,$$

then

$$\sum_{j=1}^{\infty}(f(j) - f(j+1))^{\frac{1}{2}} < \infty.$$

The last lemma we need shows that the $C^{2,\alpha}$-closeness criterion of Theorem 6.39 holds for all large enough times in an RMCF, so that Lemma 6.40 becomes applicable.

Lemma 6.41. *Let Σ_s be an RMCF with a tangent flow in $\mathcal{C}_k$. Given $\varepsilon, R > 0$, there exists $T < \infty$ so that*

- *For all $t \geq T$, there is a cylinder $\Gamma_t \in \mathcal{C}_k$ such that Σ_s is $(\varepsilon, R, C^{2,\alpha})$-close to Γ_t for each $s \in [t-1, t+1]$.*

Proof. If not, then there is a sequence of times $t_i \to \infty$ such that

(∗) For each i, there is no cylinder in $\mathcal{C}_k$ which Σ_s is $(\varepsilon, R, C^{2,\alpha})$-close to for all $s \in [t_i-1, t_i+1]$.

By the compactness theorem for rescaled Brakke flows, a subsequence of the RMCFs $\widetilde{\Sigma}^{(i)} = \{\Sigma_s : s \in [t_i-1, t_i+1]\}$ converges to a rescaled Brakke flow $\widetilde{\Gamma}$. By Theorem 6.3 and Theorem 4.1, $\widetilde{\Gamma}$ must be the rescaled Brakke flow of a unit multiplicity smooth cylinder in $\mathcal{C}_k$, which is simply a stationary RMCF (Lemma 3.21). By Brakke's regularity theorem [Bra78], the convergence $\widetilde{\Sigma}^{(i)} \to \widetilde{\Gamma}$ is smooth on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$. This contradicts (∗). □

Proof of Theorem 6.1: uniqueness of cylindrical tangent flows. Let Σ_s be an RMCF of compact, embedded hypersurfaces in $\mathbb{R}^{n+1}$, and suppose Σ_s has a tangent flow in $\mathcal{C}_k$. Let Theorem 6.39 provide ε_E, R_0 and R_1. By Lemma 6.41, there exists $T < \infty$ so that

- For all $t \geq T$, there is a cylinder $\Gamma_t \in \mathcal{C}_k$ such that Σ_s is $(\varepsilon_E, R_1, C^{2,\alpha})$-close to Γ_t for each $s \in [t-1, t+1]$.

Note that $\lim_{t\to\infty} R(\Sigma_t) = \infty$ by the monotonicity of $\mathcal{F}$ (Lemma 3.24) and the definition of $R(\Sigma_t)$. Thus, we may assume $R(\Sigma_t) \geq R_0$ for all $t \geq T$. Applying Theorem 6.39 then Lemma 6.40 yields

$$\sum_{j=1}^{\infty} (\mathcal{F}(\Sigma_j) - \mathcal{F}(\Sigma_{j+1}))^{\frac{1}{2}} < \infty.$$

Using this, the Cauchy–Schwarz inequality $\|\phi\|^2_{L^1(\Sigma_s)} \leq (4\pi)^{\frac{n}{2}} \mathcal{F}(\Sigma_s) \|\phi\|^2_{L^2(\Sigma_s)}$, and the monotonicity of $\mathcal{F}$, we compute

$$\begin{aligned}\int_1^\infty \|\phi\|_{L^1(\Sigma_s)}\, ds &\leq \sum_{j=1}^{\infty} \int_j^{j+1} (4\pi)^{\frac{n}{4}} \sqrt{\mathcal{F}(\Sigma_s)}\, \|\phi\|_{L^2(\Sigma_s)}\, ds \\ &\leq (4\pi)^{\frac{n}{4}} \sqrt{\mathcal{F}(\Sigma_1)} \sum_{j=1}^{\infty} (\mathcal{F}(\Sigma_j) - \mathcal{F}(\Sigma_{j+1}))^{\frac{1}{2}} < \infty.\end{aligned}$$

From this and Lemma B.5, it follows that the total area swept out by the RMCF within B_{R_1}, weighted by $e^{-\frac{|x|^2}{4}}$, is finite. Namely, one applies Lemma B.5 to the time intervals $[T-1, T+1]$, $[T+1, T+3]$, and so on (laying this out in full would unfortunately require excessive notation). But Theorem 6.3 says that $B_{R_1} \cap \Sigma_s$ subconverges smoothly to a cylinder along every sequence of times, so the only way to sweep out a finite area is for $B_{R_1} \cap \Sigma_s$ to converge to a unique cylinder as $s \to \infty$. Repeating this at all scales $R > R_1$ gives the uniqueness. □

Chapter 7

Recent Progress and Research Directions

In this chapter, we outline some developments which stem from, or are closely related to, the results presented in this thesis. However, we begin with a retrospective note explaining why we have already proved uniqueness of tangent flows not only for mean convex MCFs, but for 'most' MCFs.

7.1 Generic mean curvature flow

Recall from §4.1 that a shrinker with nonpositive definite stability operator L locally minimises the $\mathcal{F}$-functional among nearby hypersurfaces. In [CM12], Colding and Minicozzi used this to arrive at the following insight. Since RMCF is the negative gradient flow of $\mathcal{F}$, shrinkers that are stable in this sense are tangent flows that cannot be perturbed away by varying the initial conditions, and therefore represent singularities of a *generic* MCF. Hence, if we understand the stable shrinkers, then we understand the singularities of 'most' MCFs.

In practice, one uses a slightly different notion of stability called *entropy-stability* to quotient out by invariances of MCF under symmetries like translation, rotations and parabolic dilations. An entropy-stable shrinker is one that locally minimises the entropy functional defined in (6.1). The main result of [CM12] is a classification of entropy-stable shrinkers. Theorem 4.1 is a key step in achieving this, and the final classification is in fact the same:

Theorem 7.1 ([CM12]). *Every smooth, embedded, entropy-stable shrinker in* $\mathbb{R}^{n+1}$ *with polynomial volume growth is* $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$ *for some* $k \in \{0, \ldots, n\}$.

This shows that a generic MCF can only have the simplest shrinkers as its tangent flows, whereas the exotic shrinkers, whilst abundant in numbers, only arise as tangent flows in very special cases. Better still, since we now know that all tangent flows of the type $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$ are unique, we conclude with the following.

Corollary 7.2. *Uniqueness of tangent flows holds for all generic mean curvature flows.*

7.2 More uniqueness of tangent flows

7.2.1 Uniqueness of asymptotically conical tangent flows

Even though $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$ are the only generic singularities, other singularities are still worth studying. Perhaps the next most important class of shrinkers are the *asymptotically conical* ones. Wang [Wan16] showed that all embedded noncompact shrinkers in $\mathbb{R}^3$ are asymptotically cylindrical or conical. Moreover, asymptotically conical shrinkers in $\mathbb{R}^3$ have been explicitly constructed [Ngu14, KKM18]. It turns out, as recently proved by Chodosh and Schulze [CS21], that these are also unique (in all dimensions):

Theorem 7.3 ([CS21]). *If a unit multiplicity, asymptotically conical shrinker in $\mathbb{R}^{n+1}$ arises as a tangent flow of a compact, embedded MCF, then it is the unique tangent flow at that point.*

Despite the noncompactness, this is proved by a Łojasiewicz-type inequality which, like Theorem 5.3, comes from reducing to the finite-dimensional case. It follows from this, [Wan16], and the classification and uniqueness of generic singularities that for a compact embedded MCF in $\mathbb{R}^3$, *all* unit multiplicity tangent flows are unique.

7.2.2 Uniqueness in high codimension mean curvature flow

High codimension MCF is the MCF of n-dimensional submanifolds of $\mathbb{R}^{n+k}$, $k \geq 2$. This is defined also by (3.1) but the right-hand side is replaced by the mean curvature vector $-\mathbf{H}$. The main difficulty is the nontriviality of the normal bundle, which makes the second fundamental form (now vector-valued) very intricate with a complicated evolution equation. This causes properties like the avoidance principle and preservation of embeddedness to fail in high codimension. However, the blowup procedure and the notion of a tangent flow manage to survive. See, e.g. [Bak] for an introduction to high codimension MCF.

We can therefore study uniqueness of tangent flows in high codimension. To this end, Andrews and Baker [AB10] showed that under suitably pinched curvature conditions, all tangent flows are spheres and are therefore unique (this is similar to Huisken's classic result [Hui84]). Uniqueness of compact tangent flows also holds true in high codimension, and is (quite remarkably) proved by reproducing §5 almost verbatim. Colding and Minicozzi also generalised uniqueness of cylindrical tangent flows to high codimension [CM19b], but there were additional complications that needed entirely new arguments to surmount.[1]

It would be interesting to extend the notion of generic MCF to high codimension. This would create a meaningful intermediate goal to prove uniqueness of tangent flows for generic MCFs in high codimension, just as what was done in codimension one. To our knowledge, the closest to a genericity result in high codimension is that of Andrews, Li and Wei [ALW14]. They showed that the entropy-stable shrinkers in high codimension are also spheres, cylinders and planes, but only under an artificial 'parallel principal normal' condition.

[1] In unit codimension, Lemma 6.18 depends on the relations (2.2) which are specific to hypersurfaces.

7.2.3 More Łojasiewicz inequalities

The quantity $\tau = \frac{A}{H}$ played a prominent role in the proof of the Łojasiewicz inequalities in §6. However, the techniques used there are hard to generalise to shrinkers where H possibly vanishes, as τ would not be globally defined. This motivates the development of more widely applicable methods to prove Łojasiewicz inequalities for shrinkers. We are aware of the recent work of Zhu [Zhu20, Zhu21] and Sun–Zhu [SZ20] where Łojasiewicz inequalities were proved, respectively, for generalised cylindrical shrinkers (including immersed ones) and product shrinkers of the form $S^{k_1}_{\sqrt{2k_1}} \times S^{k_2}_{\sqrt{2k_2}}$. The approach used in these papers is a perturbative method based on Taylor expansion of the quantity $\phi = -H + \frac{\langle x, \mathbf{n} \rangle}{2}$, which in principle could be done for any shrinker. In [Zhu20], this led to an alternative proof of Theorem 6.1.

7.3 Applications – understanding the singular set

7.3.1 Structure and regularity of the singular set

We have seen that different solutions to MCF can have considerably dissimilar singular sets, e.g. Figure 1.1. It seems imaginable that singular sets of MCFs can be as wild as we like, in terms of dimension and regularity. However, Colding and Minicozzi proved that for generic MCFs (i.e. where every tangent flow is $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$), the singular set must be well-behaved:

Theorem 7.4 ([CM16a]). *For a compact embedded generic MCF in $\mathbb{R}^{n+1}$, the singular set $\mathcal{S}$ is contained in finitely many compact embedded C^1 submanifolds each of dimension $(n-1)$ together with a set of dimension at most $(n-2)$.*

We will not go into how this is proved other than mentioning that it uses the uniqueness result of Corollary 7.2. This is quite surprising since uniqueness of tangent flows is a statement about one singular point, whereas Theorem 7.4 is a claim about $\mathcal{S}$ globally.[2] The exact statement in [CM16a] is even stronger than this; it very precisely describes the structure of $\mathcal{S}$.

Theorem 7.4 has a striking corollary. To understand this, we need to mention that Brakke flow has a built-in capability to flow past singularities. In Brakke flow, a smooth hypersurface remains smooth till the first singular time (it coincides with MCF), but afterwards there are usually intractable changes in topology and regularity. However, things are controllable in low dimensions:

Corollary 7.5 ([CM16a]). *For a generic MCF in $\mathbb{R}^3$ or $\mathbb{R}^4$ (where the Brakke flow is used to flow past singularities), almost every timeslice is a smooth hypersurface, and any connected subset of the space-time singular set is completely contained in a timeslice.*

Aside from Brakke flow, there are other weak formulations of MCF that admit a mechanism of flowing past singularities. Some of these include the level-set flow, the viscosity approach and the shadow flow; see [ACGL20, §6.9] for a discussion as well as the references therein. A contentious issue is deciding which formulation is the best; we would prefer one with a high degree of regularity and generality, and perhaps one that is in a way canonically associated to smooth MCF. Results like Theorem 7.4 and Corollary 7.5 could be used to settle this debate, or at

[2] Between these extremes lies the result of [CIM15], which says that in a neighbourhood of a point where there is a cylindrical tangent flow, all blowups are cylindrical. Theorem 6.3 is a special case of this.

least provide a case for using certain weak formulations over others. It turns out that uniqueness of tangent flows also has regularity implications for the level-set flow [CM16b, CM18].

7.3.2 Mean curvature flow with surgery

Under MCF with surgery, a hypersurface evolves by MCF until a singularity is imminent. One then stops the flow, performs surgery on the hypersurface to avert the singularity (see Figure 7.1), then continues the flow. This is repeated until the hypersurface is in a desired form, typically a union of (S^n)'s and $(S^{n-1} \times S^1)$'s. The motivation for this comes from the Ricci flow, where surgery procedures were vital to Perelman's proof of the Poincaré and geometrisation conjectures.

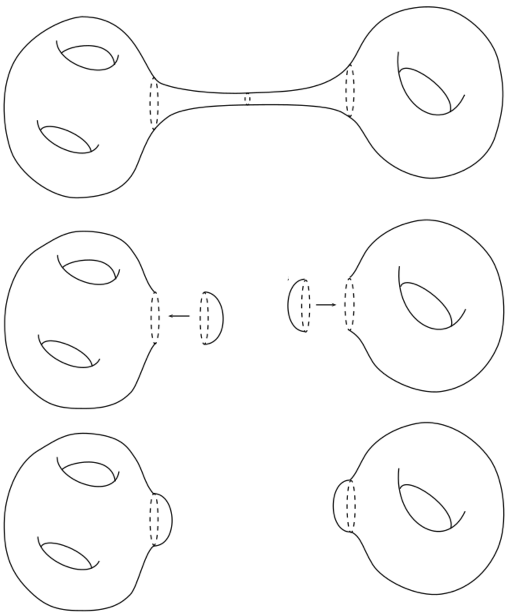

Figure 7.1: Performing surgery on the hypersurface to remove a region where a singularity would otherwise develop under MCF. Adapted from [ACGL20].

MCF with surgery is a highly ambitious task which has only succeeded in a few cases so far, most notably for 2-convex hypersurfaces in $\mathbb{R}^{n+1}$ (with $n \geq 3$) [HS09] and for mean convex hypersurfaces in $\mathbb{R}^3$ [BH16, HK17]. See also [Ngu20] for some progress in high codimension. A major difficulty is understanding what the singular set looks like so that we know exactly where surgery needs to be done; results like Theorem 7.4 and Corollary 7.5 could be valuable here. In addition, uniqueness of tangent flows tells us that singularities in MCF (morally) form at the rate of magnification, i.e. the RMCF rescaling rate. This could play a role in deciding on the correct time to pause the flow and undertake the surgery.

Appendix A

The Euler–Lagrange Functional and its Linearisation

In this appendix, (M, g) is a compact Riemannian n-manifold with Levi-Civita connection ∇. We first derive expressions for the Euler–Lagrange functional of an integral map $C^1(M) \to \mathbb{R}$ and its linearisation. We then prove some estimates that are used in the main text.

A.1 Existence and uniqueness of the Euler–Lagrange functional

Consider a functional $\mathcal{E} : C^1(M) \to \mathbb{R}$ of the form

$$\mathcal{E}(u) = \int_M E(p, u(p), \nabla u(p)),$$

where $E = E(p, q, z)$ is a smooth function of $p \in M$, $q \in \mathbb{R}$ and $z \in T_pM$, and the integration takes place with respect to the Riemannian measure μ associated to (M, g).

Recall that the Euler–Lagrange functional $\mathcal{M} : C^2(M) \to C^0(M)$ is defined by requiring

$$-\langle \mathcal{M}u, v\rangle_{L^2} = \frac{d}{ds}\Big|_{s=0} \mathcal{E}(u + sv), \qquad \forall u, v \in C^2(M).$$

Let us compute this explicitly in local coordinates. Choosing an orthonormal tangent frame $\{\partial_i\}_{i=1}^n$ around a point p, any $u \in C^2(M)$ is locally written $\nabla u = (\nabla_i u)\partial_i$. Write

$$E(p, u(p), \nabla u(p)) = \widetilde{E}(p, u(p), \nabla_1 u(p), \ldots, \nabla_n u(p)),$$

where $\widetilde{E}$ is smooth. Subsequently we abbreviate $(\nabla_1 u(p), \ldots, \nabla_n u(p))$ as $\nabla \widetilde{u}$, and suppress p-

dependences. Subscripts on functions denote partial derivatives. Then

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\mathcal{E}(u+sv) &= \int_M \frac{d}{ds}\Big|_{s=0}\widetilde{E}(p,u+sv,\nabla\widetilde{u}+s\nabla\widetilde{v})\\ &= \int_M v\widetilde{E}_q(p,u,\nabla\widetilde{u}) + (\nabla_i\widetilde{v})\widetilde{E}_{z_i}(p,u,\nabla\widetilde{u})\\ &= \int_M \left\{\sum_{i=1}^n \Big(\nabla_i(v\widetilde{E}_{z_i}(p,u,\nabla\widetilde{u})) - v\nabla_i\widetilde{E}_{z_i}(p,u,\nabla\widetilde{u})\Big) + v\widetilde{E}_q(p,u,\nabla\widetilde{u})\right\}\\ &= \int_M v\left\{-\sum_{i=1}^n \Big(\widetilde{E}_{z_i}(p,u,\nabla\widetilde{u}) + \nabla_i\widetilde{E}_{z_i}(p,u,\nabla\widetilde{u})\Big) + \widetilde{E}_q(p,u,\nabla\widetilde{u})\right\}\end{aligned}$$

This shows that $\mathcal{M}(u) \in C^0(M)$ is given locally by

$$\begin{aligned}\mathcal{M}(u)(p) &= \sum_{i=1}^n \Big(\nabla_i\widetilde{E}_{z_i}(p,u,\nabla\widetilde{u}) + \widetilde{E}_{z_i}(p,u,\nabla\widetilde{u})\Big) - \widetilde{E}_q(p,u,\nabla\widetilde{u})\\ &= \sum_{i,j=1}^n \widetilde{E}_{z_iz_j}(p,u,\nabla\widetilde{u})\nabla_i\nabla_j u + f(p,u,\nabla\widetilde{u}),\end{aligned} \tag{A.1}$$

where $f(p,q,z)$ is a smooth real-valued function. Note that this is quasilinear: only the dependence on second-order partial derivatives of u is necessarily linear. The linearisation of $\mathcal{M}$ at u is a linear map $L_u : C^2(M) \to C^0(M)$ defined by

$$L_u w = \frac{d}{ds}\Big|_{s=0}\mathcal{M}(u+sw) = \sum_{i,j=1}^n \widetilde{E}_{z_iz_j}(p,u,\nabla\widetilde{u})\nabla_i\nabla_j w + R, \tag{A.2}$$

where R has linear dependence on $\widetilde{w}$, $\nabla\widetilde{w}$. In (2.12), we showed that L_u is symmetric. Now if we impose that for some $C > 0$

$$\frac{d^2}{ds^2}\Big|_{s=0}E(p,0,sz) \geq C|z|^2, \qquad \forall z \in T_pM,$$

then we can check that $L = L_0$ is uniformly elliptic. Indeed, let $\xi = \xi_i\varepsilon^i \in T_p^*M$, where $\{\varepsilon^i\}_{i=1}^n$ is the dual coframe of $\{\partial_i\}_{i=1}^n$. Then from (A.2) we have

$$\sum_{i,j=1}^n \widetilde{E}_{z_iz_j}(p,0,0)\xi_i\xi_j = \frac{d^2}{ds^2}\Big|_{s=0}\widetilde{E}(p,0,s\xi_1,\ldots,s\xi_n) = \frac{d^2}{ds^2}\Big|_{s=0}E(p,0,s\xi) \geq C|\xi|^2,$$

which is the definition of uniform ellipticity.

A.2 Some estimates

In this section, we allow $\mathcal{M} : C^2(M) \to C^0(M)$ to be any map which is locally of the form

$$\mathcal{M}(u)(p) = \Phi^{ij}(p,u(p),\nabla u(p))u_{ij}(p) + f(p,u(p),\nabla u(p)),$$

where Φ^{ij} and f are smooth functions of (p,q,z) where $p \in M$, $q \in \mathbb{R}$ and $z \in T_pM$, and u_{ij} means $\nabla_i\nabla_j u$. In particular, this is the form of $\mathcal{M}$ in the previous section (see (A.1)). The linearisation of $\mathcal{M}$ at u is

$$L_u v = \frac{d}{ds}\Big|_{s=0}\mathcal{M}(u+sv) = u_{ij}\left(\Phi^{ij}_q v + \Phi^{ij}_{z_\alpha}v_\alpha\right) + \Phi^{ij}v_{ij} + f_q v + f_{z_\beta}v_\beta, \tag{A.3}$$

where f, Φ and its derivatives are evaluated at $(p, u(p), \nabla u(p))$.

For the next lemma, we omit the proof which is a simple calculation using the fundamental theorem of calculus and the chain rule.

Lemma A.1 ([CM19a])**.** *If f is a C^1 function of (p, q, z), and $u, v \in C^1(M)$, then*

$$|f(p, u(p), \nabla u(p)) - f(p, v(p), \nabla v(p))| \leq C_f(|u(p) - v(p)| + |\nabla u(p) - \nabla v(p)|),$$

where $\overline{C}_f = \overline{C}_f(p) = \sup\{|f_q| + |f_{y_\alpha}| \mid |q| + |z| \leq \|u\|_{C^1} + \|v\|_{C^1}\}$.

Proposition A.2 ([CM15])**.** *The linearisation L_u deviates at most quadratically from $\mathcal{M}$ in the sense that at each point $p \in M$, we have for all $u, v \in C^2(M)$ that*

$$\mathcal{M}(u + v) - \mathcal{M}(u) = L_u v + R(u, v, p),$$

where the remainder term $R(u, v, p)$ satisfies (at $p \in M$)

$$\begin{aligned} |R| &\leq C_1(|v| + |\nabla v|)^2 + C_2(|v| + |\nabla v|)|\nabla^2 v|, \\ C_1 &= |\Phi_q^{ij}| + |\Phi_{z_\alpha}^{ij}| + \overline{C}_{\Phi^{ij}}, \\ C_2 &= |u_{ij}|\overline{C}_{\Phi_q^{ij}} + |u_{ij}|\overline{C}_{\Phi_{z_\alpha}^{ij}} + \overline{C}_{f_q} + \overline{C}_{f_{z_\beta}}. \end{aligned}$$

Here the suprema defining the $\overline{C}$ terms are taken over $|q| + |z| \leq 2\|u\|_{C^1} + \|v\|_{C^1}$.

Proof. Using (A.3) and Lemma A.1 we compute (at p)

$$\begin{aligned} |L_{u+w}v - L_u v| &= |(u_{ij} + w_{ij})\left(\Phi_q^{ij}(p, u + w, \nabla u + \nabla w)v + \Phi_{z_\alpha}^{ij}(p, u + w, \nabla u + \nabla w)v_\alpha\right) \\ &\quad - u_{ij}(\Phi_q^{ij}(p, u, \nabla u)v + \Phi_{z_\alpha}^{ij}(p, u, \nabla u)v_\alpha) \\ &\quad + \Phi^{ij}(p, u + w, \nabla u + \nabla w)v_{ij} - \Phi^{ij}(p, u, \nabla u)v_{ij} \\ &\quad + f_q(p, u + w, \nabla u + \nabla w)v - f_q(p, u, \nabla u)v \\ &\quad + f_{z_\beta}(p, u + w, \nabla u + \nabla w)v_\beta - f_{z_\beta}(p, u, \nabla u)v_\beta| \\ &\leq |u_{ij}|\overline{C}_{\Phi_q^{ij}}(|w| + |\nabla w|)|v| + |u_{ij}|\overline{C}_{\Phi_{z_\alpha}^{ij}}(|w| + |\nabla w|)|v_\alpha| \\ &\quad + |w_{ij}||\Phi_q^{ij}||v| + |w_{ij}||\Phi_{z_\alpha}^{ij}||v_\alpha| + \overline{C}_{\Phi^{ij}}(|w| + |\nabla w|)|v_{ij}| \\ &\quad + \overline{C}_{f_q}(|w| + |\nabla w|)|v| + \overline{C}_{f_{z_\beta}}(|w| + |\nabla w|)|v_\beta| \\ &\leq \left\{|\Phi_q^{ij}| + |\Phi_{z_\alpha}^{ij}|\right\}(|v| + |\nabla v|)|\nabla^2 w| + \overline{C}_{\Phi^{ij}}(|w| + |\nabla w|)|\nabla^2 v| \\ &\quad + \left\{|u_{ij}|\overline{C}_{\Phi_q^{ij}} + |u_{ij}|\overline{C}_{\Phi_{z_\alpha}^{ij}} + \overline{C}_{f_q} + \overline{C}_{f_{z_\beta}}\right\}(|w| + |\nabla w|)(|v| + |\nabla v|), \end{aligned} \tag{A.4}$$

where the suprema defining the $\overline{C}$ terms are taken over $|q| + |z| \leq 2\|u\|_{C^1} + \|w\|_{C^1}$ over the point p. Using the fundamental theorem of calculus, we have (again at p) that

$$\mathcal{M}(u + v) - \mathcal{M}(u) = \int_0^1 \left(\frac{d}{dt}\Big|_{t=0} \mathcal{M}(u + tv)\right) dt = \int_0^1 L_{u+tv} v \, dt = L_u v + R, \tag{A.5}$$

where $R = \int_0^1 (L_{u+tv} - L_u v)\, dt$. Using (A.4) we bound $|R|$ by

$$\begin{aligned} |R| &\leq \sup_{t \in [0,1]} |L_{u+tv} v - L_u v| \\ &\leq \left\{|\Phi_q^{ij}| + |\Phi_{z_\alpha}^{ij}| + \overline{C}_{\Phi^{ij}}\right\}(|v| + |\nabla v|)|\nabla^2 v| \\ &\quad + \left\{|u_{ij}|\overline{C}_{\Phi_q^{ij}} + |u_{ij}|\overline{C}_{\Phi_{z_\alpha}^{ij}} + \overline{C}_{f_q} + \overline{C}_{f_{z_\beta}}\right\}(|v| + |\nabla v|)^2, \end{aligned}$$

which is the proposition. □

In the main text, we use the following variation of the above result:

Proposition A.3. *Suppose* $\|u\|_{C^2}, \|v\|_{C^2} < 1$*, and let* $L = L_0$*. Then*

$$\mathcal{M}u - \mathcal{M}v = L(u-v) + a^{ij}(u-v)_{ij} + b^{\alpha}(u-v)_{\alpha} + c(u-v),$$

where a^{ij}*,* b^{α} *and* c *are functions of* $p \in M$ *with*

$$\sup_M (|a^{ij}| + |b^{\alpha}| + |c|) \le C(\|u\|_{C^2} + \|v\|_{C^2}),$$

and C *depends only on* n *and the form of* $\mathcal{M}$*.*

Proof. From (A.5), we have at each point $p \in M$ that

$$\mathcal{M}u - \mathcal{M}v = L_v(u-v) + R = L(u-v) + (L_v(u-v) - L(u-v)) + R. \tag{A.6}$$

We will bound $L_v(u-v) - L(u-v)$ and R. From Proposition A.2, we already know that

$$|R| \le C_1(|u-v| + |\nabla(u-v)|)^2 + C_2(|u-v| + |\nabla(u-v)|)|\nabla^2(u-v)|.$$

By inspecting the forms of C_1, C_2, we have $C_1 + C_2 \le C(1 + \|v\|_{C^2}) \le C$, as $\|v\|_{C^2} < 1$. Thus

$$\begin{aligned} |R| &\le C(|u-v| + |\nabla(u-v)|)(|u-v| + |\nabla(u-v)| + |\nabla^2(u-v)|) \\ &\le C(\|u\|_{C^1} + \|v\|_{C^1})(|u-v| + |\nabla(u-v)| + |\nabla^2(u-v)|). \end{aligned} \tag{A.7}$$

On the other hand, using (A.4) we have

$$\begin{aligned} |L_v(u-v) - L(u-v)| &\le C(|u-v| + |\nabla(u-v)|)|\nabla^2 v| + C(|v| + |\nabla v|)|\nabla^2(u-v)| \\ &\quad + C(|v| + |\nabla v|)(|u-v| + |\nabla(u-v)|) \\ &\le C\|v\|_{C^2}(|u-v| + |\nabla(u-v)| + |\nabla^2(u-v)|). \end{aligned} \tag{A.8}$$

By (A.7) and (A.8) we see that we can write

$$L_v(u-v) - L(u-v) + R = a^{ij}(u-v)_{ij} + b^{\alpha}(u-v)_{\alpha} + c(u-v),$$

where a^{ij}, b^{α} and c are functions of $p \in M$ with

$$\sup_M (|a^{ij}| + |b^{\alpha}| + |c|) \le C(\|u\|_{C^2} + \|v\|_{C^2}).$$

Together with (A.6), this gives the proposition. □

Appendix B

Graphs Over Hypersurfaces

This appendix gathers some calculations for hypersurfaces written as normal graphs over an embedded hypersurface $\Sigma \subset \mathbb{R}^{n+1}$. We follow Appendix A of [CM15] and Appendix A of [CM19a], although some of our calculations do not appear there (but are needed in the main text).

Let Σ have normal injectivity radius δ. Then every smooth function $u : \Sigma \to \mathbb{R}$ with $\|u\|_{C^0} \leq \delta$ gives rise to a hypersurface $\Sigma_u = \mathrm{graph}_\Sigma(u)$. We use x and y to denote generic points on Σ and Σ_u respectively. Their correspondence is given by $y = x + u(x)\mathbf{n}(x)$ where $\mathbf{n}$ is the normal to Σ.

B.1 Key graphical quantities

For $x \in \Sigma$, let $\mathbf{n}_u(x)$ be the unit normal to Σ_u at $y = x + u(x)\mathbf{n}(x)$, that is $\mathbf{n}_u(x) = \mathbf{n}^{\Sigma_u}(y)$. The following is Lemma A.3 and Corollary A.30 of [CM15], and expresses the following functions on Σ in terms of u.

- The mean curvature $H_u(x)$ on Σ_u at $x + u(x)\mathbf{n}(x)$, that is $H_u(x) = H^{\Sigma_u}(y)$;
- The relative volume element $\nu_u(x)$ defined by requiring that $\int_{\Sigma_u} f(y) = \int_\Sigma f(y)\nu_u(x)$ for all test functions $f : \Sigma_u \to \mathbb{R}$;
- The support function $\eta_u(x) = \langle y, \mathbf{n}_u(x)\rangle$;
- The speed function $w_u(x) = \langle \mathbf{n}(x), \mathbf{n}_u(x)\rangle^{-1}$.

Lemma B.1. *There are functions w, ν and η of $(x, q, z) \in \Sigma \times \mathbb{R} \times T_x\Sigma$ which are smooth for $|q| < \delta$ such that*

$$w_u(x) = w(x, u(x), \nabla u(x)), \quad \nu_u(x) = \nu(x, u(x), \nabla u(x)), \quad \eta_u(x) = \eta(x, u(x), \nabla u(x)).$$

Writing $B(x, q) = \mathrm{Id} - qA(x)$ where A is the matrix of the second fundamental form for Σ at x, these functions are given by

$$\begin{aligned} w(x, q, z) &= \sqrt{1 + |B(x, q)^{-1}(z)|^2}, \\ \nu(x, q, z) &= w(x, q, z) \det B(x, q), \\ \eta(x, q, z) &= \frac{\langle x, \mathbf{n}(x)\rangle + q - \langle x, B(x, q)^{-1}(z)\rangle}{w(x, q, z)}. \end{aligned}$$

Secondly, the following table gathers some basic values and derivatives of these functions.

Function	*Value at* $(x,0,0)$	∂_q *at* $(x,0,0)$	∂_{z_j} *at* $(x,0,0)$
w	1	0	0
ν	1	$H(x)$	0
η	$\langle x, \mathbf{n}(x)\rangle$	1	$-x_j$
$\partial_{z_i}\nu$	0	0	δ_{ij}
$\partial_q\nu$	$H(x)$	$H^2(x) - \lvert A\rvert^2(x)$	0

Finally, the function H_u is given by

$$H_u(x) = \frac{w}{\nu}\left\{\partial_q\nu - \partial_{x_i}\partial_{z_i}\nu - (\partial_q\partial_{z_i}\nu)\nabla_i u(x) - (\partial_{z_i}\partial_{z_j}\nu)\nabla_i\nabla_j u(x)\right\},$$

where w, ν and their derivatives are evaluated at $(x, u(x), \nabla u(x))$.

B.2 The Euler–Lagrange functional of $\mathcal{F}_\Sigma$

Recall from §4.1 that $\mathcal{F}_\Sigma : C^1(\Sigma)\cap B_\delta(0) \to \mathbb{R}$ is given by

$$\mathcal{F}_\Sigma(u) = (4\pi)^{-\frac{n}{2}}\int_{\Sigma_u} e^{-\frac{|y|^2}{4}} = (4\pi)^{-\frac{n}{2}}\int_\Sigma e^{-\frac{|x+u(x)\mathbf{n}(x)|^2}{4}}\nu_u(x), \tag{B.1}$$

where the second equality is the definition of ν_u. The small C^1 norm ensures that $\nu_u \approx 1$ is bounded and so by (B.1), $\mathcal{F}_\Sigma(u)$ is close to $\mathcal{F}_\Sigma(0)$. We will compute the Euler–Lagrange functional of $\mathcal{F}_\Sigma$ with respect to the Riemannian and Gaussian measures on Σ. Denote these by $\mathcal{M}_\Sigma$ and $\mathcal{M}^\nu_\Sigma$ respectively. Both are operators from $C^2(\Sigma)\cap B_\delta(0)$ to $C^0(\Sigma)$, defined by requiring that

$$-\int_\Sigma v\mathcal{M}_\Sigma(u) = \frac{d}{ds}\Big|_{s=0}\mathcal{F}_\Sigma(u+sv) = -(4\pi)^{-\frac{n}{2}}\int_\Sigma v\mathcal{M}^\nu_\Sigma(u)e^{-\frac{|x|^2}{4}}.$$

Proposition B.2. *Explicit formulae for $\mathcal{M}_\Sigma$ and $\mathcal{M}^\nu_\Sigma$ are given by*

$$\mathcal{M}_\Sigma(u) = -(4\pi)^{-\frac{n}{2}}\left(H_u - \frac{\langle x+u(x)\mathbf{n}(x), \mathbf{n}_u\rangle}{2}\right)\langle \mathbf{n}, \mathbf{n}_u\rangle\, e^{-\frac{|x+u(x)\mathbf{n}(x)|^2}{4}}, \tag{B.2}$$

$$\mathcal{M}^\nu_\Sigma(u) = -\left(H_u - \frac{1}{2}\eta_u\right)(\det B(x,u))e^{-\frac{u^2+2u\langle x,\mathbf{n}\rangle}{4}}. \tag{B.3}$$

Proof. Let $u, v \in C^2(\Sigma)\cap B_\delta(0)$. First note that by a geometric argument,

$$\frac{d}{ds}\Big|_{s=0}\mathcal{F}_\Sigma(u+sv) = \frac{d}{ds}\Big|_{s=0}\mathcal{F}_{\Sigma_u}(s\left\langle v_*\mathbf{n}_*, \mathbf{n}^{\Sigma_u}\right\rangle),$$

where $v_* : \Sigma_u \to \mathbb{R}$ is given by $v_*(y) = v(x)$, similarly for $\mathbf{n}_*$, and $\mathbf{n}^{\Sigma_u}$ is the normal to Σ_u. Repeating the computation (4.3) then pulling back from Σ_u to Σ, we get

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\mathcal{F}_\Sigma(u+sv) &= (4\pi)^{-\frac{n}{2}}\int_{\Sigma_u}\left(H^{\Sigma_u} - \frac{\left\langle y, \mathbf{n}^{\Sigma_u}\right\rangle}{2}\right)v_*\left\langle \mathbf{n}_*, \mathbf{n}^{\Sigma_u}\right\rangle e^{-\frac{|y|^2}{4}} \\ &= (4\pi)^{-\frac{n}{2}}\int_\Sigma\left(H_u(x) - \frac{\langle x+u(x)\mathbf{n}(x), \mathbf{n}_u\rangle}{2}\right)v(x)\langle \mathbf{n}, \mathbf{n}_u\rangle\, e^{-\frac{|y|^2}{4}}\nu_u(x)\end{aligned} \tag{B.4}$$

which gives (B.2). Note that Lemma B.1 gives $\langle \mathbf{n}, \mathbf{n}_u \rangle \nu_u = \frac{\nu_u}{w_u} = \det B(x,u)$. Also, $|x+u(x)\mathbf{n}(x)|^2 = |x|^2 + u^2 + 2u\langle x, \mathbf{n}\rangle$ and $\langle x+u(x)\mathbf{n}(x), \mathbf{n}_u\rangle = \eta_u$ by definition. Using these in (B.4), we have

$$\frac{d}{ds}\Big|_{s=0} \mathcal{F}_\Sigma(u+sv) = (4\pi)^{-\frac{n}{2}} \int_\Sigma \left(H_u(x) - \frac{1}{2}\eta_u(x)\right) v(x)(\det B(x,u))e^{-\frac{u^2+2u\langle x,\mathbf{n}\rangle}{4}} e^{-\frac{|x|^2}{4}},$$

from which (B.3) follows. □

Proof of Proposition 4.5. The $L^2(\nu)$ Euler–Lagrange functional from the statement of the proposition refers to $\mathcal{M}^\nu_\Sigma$ defined above. By Proposition B.2, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{M}^\nu_\Sigma(\varepsilon u) = -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(H_{\varepsilon u} - \frac{1}{2}\eta_{\varepsilon u}\right)(\det B(x,\varepsilon u))e^{-\frac{\varepsilon^2u^2+2\varepsilon u\langle x,\mathbf{n}\rangle}{4}}. \tag{B.5}$$

Since $\eta_0(x) = \langle x, \mathbf{n}\rangle$ (see the table in Lemma B.1) and Σ is a shrinker by assumption, we have $H_0 - \frac{1}{2}\eta_0 = 0$. So the only nonzero term in (B.5) is the term where we differentiate $H_{\varepsilon u} - \frac{1}{2}\eta_{\varepsilon u}$. By the evolution equations for H and $\mathbf{n}$ in Lemma 3.12 (with $f = u$ there), we compute

$$\begin{aligned} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{M}^\nu_\Sigma(\varepsilon u) &= -(\det B(x,0))\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(H_{\varepsilon u} - \frac{1}{2}\eta_{\varepsilon u}\right) \\ &= -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(H_{\varepsilon u} - \frac{1}{2}\langle x+\varepsilon u\mathbf{n}, \mathbf{n}_{\varepsilon u}\rangle\right) \\ &= -\left(-\Delta u - |A|^2u - \frac{1}{2}\langle u\mathbf{n}, \mathbf{n}\rangle + \frac{1}{2}\langle x, \nabla u\rangle\right) \\ &= Lu. \end{aligned} \tag{B.6}$$

That is, L is the linearisation of $\mathcal{M}^\nu_\Sigma$ at zero, which is the first claim of the proposition. For the second claim, let $\psi \in C^2(\Sigma)$. Then (B.3) and (B.6) together give

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} \mathcal{F}_\Sigma(\varepsilon\psi) = (4\pi)^{-\frac{n}{2}}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_\Sigma \psi\mathcal{M}^\nu_\Sigma(\varepsilon\psi)e^{-\frac{|x|^2}{4}} = (4\pi)^{-\frac{n}{2}}\int_\Sigma \psi L\psi e^{-\frac{|x|^2}{4}}.$$

This completes the proof. □

B.3 Graphical solutions of rescaled mean curvature flow

Let $u : \Sigma \times [s_1, s_2] \to \mathbb{R}$ be a one-parameter family of smooth functions on Σ. Assume that $\|u(\cdot,s)\|_{C^0} < \delta$ for each $s \in [s_1,s_2]$, so that $\Sigma_s = \mathrm{graph}_\Sigma(u(\cdot,s))$ is a smooth hypersurface for each s. We will write $u(s)$ to mean $u(\cdot,s)$. The next lemma gives an evolution equation for u when Σ_s evolves by the rescaled mean curvature flow (RMCF, see Definition 3.20). By Lemma 3.21, this means that the variation field X_s on Σ_s has

$$\langle X_s, \mathbf{n}^{\Sigma_s}\rangle = -H^{\Sigma_s} + \frac{1}{2}\langle y, \mathbf{n}^{\Sigma_s}\rangle. \tag{B.7}$$

Proposition B.3. *The hypersurfaces $\Sigma_s = \mathrm{graph}_\Sigma(u(s))$ evolve by RMCF if and only if*

$$\frac{\partial u}{\partial s}(x,s) = w_{u(s)}(x)\left(\frac{1}{2}\eta_{u(s)}(x) - H_{u(s)}(x)\right).$$

This is a quasilinear parabolic equation when $u(s)$ has sufficiently small C^1 norm for each s.

Proof. At time s, the variation field and unit normal for Σ_s are $\frac{\partial u}{\partial s}\mathbf{n}$ and $\mathbf{n}_{u(s)}$ respectively. Thus,

$$\begin{aligned}\frac{1}{w}\frac{\partial u}{\partial s} &= \langle \mathbf{n}, \mathbf{n}_{u(s)} \rangle \frac{\partial u}{\partial s} = \left\langle \frac{\partial u}{\partial s}\mathbf{n}, \mathbf{n}_{u(s)} \right\rangle = -H_{u(s)} + \frac{1}{2}\langle x + u(x,s)\mathbf{n}(x), \mathbf{n}_{u(s)} \rangle \\ &= -H_{u(s)} + \frac{1}{2}\eta_{u(s)},\end{aligned} \tag{B.8}$$

where the third equality is (B.7). Multiplying by w gives the claimed formula. To see that this is quasilinear parabolic, use the formula for H_u in Lemma B.1 to write

$$\begin{aligned}\frac{\partial u}{\partial s}(x,s) &= w_{u(s)}\left(\frac{1}{2}\eta_{u(s)} - \frac{w}{\nu}\left\{\partial_q\nu - \partial_{x_i}\partial_{z_i}\nu - (\partial_q\partial_{z_i}\nu)\nabla_i u(x,s) - (\partial_{z_i}\partial_{z_j}\nu)\nabla_i\nabla_j u(x,s)\right\}\right) \\ &= \frac{w^2}{\nu}(\partial_{z_i}\partial_{z_j}\nu)\nabla_i\nabla_j u(x,s) + (\text{terms in } x, u, \nabla u).\end{aligned}$$

Here w, ν and $\partial_{z_i}\partial_{z_j}\nu$ are all evaluated at $(x, u(x,s), \nabla u(x,s))$. Let the right-hand side be Lu. Then L is evidently quasilinear. Since ν is smooth, the leading coefficients $a^{ij} = \frac{w^2}{\nu}(\partial_{z_i}\partial_{z_j}\nu)$ of L are symmetric in i and j. When $u = 0$, the table in Lemma B.1 gives $a^{ij} = \delta_{ij}$, so if u has small enough C^1 norm then the matrix (a^{ij}) remains positive definite by the smoothness of w, ν and $\partial_{z_i}\partial_{z_j}\nu$. Therefore, L is uniformly elliptic provided that $\|u\|_{C^1} < \sigma$ say. It follows that $\frac{\partial u}{\partial s} = Lu$ is quasilinear parabolic, which is the second claim of the proposition. □

Corollary B.4. *Let u be a graphical solution to RMCF in the above sense. Given $\sigma_0 > 0$ sufficiently small, there exists a positive constant $C = C(n, \sigma_0)$ such that if $\|u(s)\|_{C^1} \le \sigma_0$, then*

$$\left|\left\langle \frac{\partial u}{\partial s}\mathbf{n}, \mathbf{n}_{u(s)} \right\rangle\right| \ge C\left|\frac{\partial u}{\partial s}\right|.$$

Proof. From (B.8), we know that

$$\left|\left\langle \frac{\partial u}{\partial s}\mathbf{n}, \mathbf{n}_{u(s)} \right\rangle\right| = \frac{1}{|w|}\left|\frac{\partial u}{\partial s}\right|,$$

where w is evaluated at $(x, u(x,s), \nabla u(x,s))$. Since w is smooth and is equal to one when $u = 0$, taking σ_0 sufficiently small means that the bound $\|u\|_{C^1} \le \sigma_0$ ensures $|w| \le C$ say, where C depends on σ_0 and the form of w, hence on n, σ_0. The corollary follows. □

Lastly, we need a bound for the area swept out by an almost cylindrical RMCF when it is graphical over a fixed cylinder. Here $\mathcal{C}_k$ is the set of all rotations in $\mathbb{R}^{n+1}$ of the cylinder $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$.

Lemma B.5. *Given $\sigma_0 > 0$ sufficiently small, there exists $C = C(n, \sigma_0)$ so that if $\Sigma \in \mathcal{C}_k$ and $u(s) \in C^1(B_R \cap \Sigma)$ is a graphical solution to RMCF for $s \in [s_1, s_2]$ with $\|u(s)\|_{C^1} \le \sigma_0$, then*

$$\int_{B_R\cap\Sigma} |u(x,s_2) - u(x,s_1)| e^{-\frac{|x|^2}{4}} \le C\int_{s_1}^{s_2}\left(\int_{\Sigma_{u(s)}}\left|\frac{\langle y, \mathbf{n}\rangle}{2} - H\right| e^{-\frac{|y|^2}{4}}\right) ds.$$

Proof. Since Σ is a shrinker by Theorem 4.1, for $x \in \Sigma$ we have $\frac{\langle x, \mathbf{n}\rangle}{2} = H = \sqrt{\frac{k}{2}}$ and so

$$|y|^2 = |x + u(x)\mathbf{n}(x)|^2 = |x|^2 + u^2 + 2u\langle x, \mathbf{n}\rangle = |x|^2 + u^2 + 2\sqrt{2k}u.$$

This together with Proposition B.3 gives (for $s \in [s_1, s_2]$)

$$\int_{B_R \cap \Sigma} \left|\frac{\partial u}{\partial s}\right| e^{-\frac{|x|^2}{4}} = \int_{B_R \cap \Sigma} |w_{u(s)}| \left|\frac{1}{2}\eta_{u(s)} - H_{u(s)}\right| e^{-\frac{|y|^2}{4} + \frac{u^2 + 2\sqrt{2k}u}{4}}.$$

By Lemma B.1, w_u and ν_u are both one when $u = 0$, so the C^1 bound on $u(s)$ yields positive upper and lower bounds on $w_{u(s)}$, $\nu_{u(s)}$ depending on n and σ_0. It follows that

$$\begin{aligned}\int_{B_R \cap \Sigma} \left|\frac{\partial u}{\partial s}\right| e^{-\frac{|x|^2}{4}} &\leq C \int_{B_R \cap \Sigma} \left|\frac{1}{2}\eta_{u(s)} - H_{u(s)}\right| e^{-\frac{|y|^2}{4}} \\ &\leq C \int_{\Sigma} \left|\frac{1}{2}\eta_{u(s)} - H_{u(s)}\right| \nu_u e^{-\frac{|y|^2}{4}} \\ &= C \int_{\Sigma_{u(s)}} \left|\frac{\langle y, \mathbf{n}\rangle}{2} - H\right| e^{-\frac{|y|^2}{4}}.\end{aligned}$$

It follows by Fubini's theorem that

$$\begin{aligned}\int_{B_R \cap \Sigma} |u(x, s_2) - u(x, s_1)| e^{-\frac{|x|^2}{4}} &\leq \int_{B_R \cap \Sigma} \left(\int_{s_1}^{s_2} \left|\frac{\partial u}{\partial s}(x, s)\right| ds\right) e^{-\frac{|x|^2}{4}} \\ &\leq C \int_{s_1}^{s_2} \left(\int_{\Sigma_{u(s)}} \left|\frac{\langle y, \mathbf{n}\rangle}{2} - H\right| e^{-\frac{|y|^2}{4}}\right) ds,\end{aligned}$$

giving the lemma. □

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