Characteristic classes for the differential geometer

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How can vector bundles be distinguished from one another? Answers to this question date back to the 1930s, by the work of Stiefel, Whitney, Pontryagin, Chern and others. They found that the structure of vector bundles can be encoded in cohomology classes known as *characteristic classes*, and these enable us to tell apart non-isomorphic bundles.

An important line of development, due to Chern and Weil in the 1940s, is the construction of characteristic classes from differential geometry, with de Rham cohomology as the cohomology theory of choice. Section 1 of this essay introduces this now-called *Chern–Weil construction*. In Section 2, we use this procedure to construct the *Chern classes* of a complex vector bundle, compute the Chern classes of $T\mathbb{CP}^n$, and finally deduce some topological results. In Section 3, we explain how power series can be used to generate characteristic classes, and give a few examples of classes arising in this way. This allows us to get a glimpse into index theory which remains an active arena for research. In particular, we sketch a the proof of the Hirzebruch signature theorem.

A general treatment of characteristic classes requires a decent amount of algebraic topology. Our avoidance of algebraic topology is somewhat a disservice to the subject; on the other hand, this makes our exposition accessible to someone who is familiar with differential geometry, but only knows de Rham cohomology as far as algebraic topology goes (e.g. me at the time of writing). The main references consulted while learning this material were [Ath18, §2], [MS74] and [Nic07, §8].

1 Chern–Weil theory

1.1 Invariant polynomials

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra with scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Our starting point is the space $I^k(G)$ which consists of symmetric, multilinear maps

$$\varphi : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k \text{ times}} \to \mathbb{K}$$

which are invariant under the adjoint action of G on $\mathfrak{g}$ (or simply ad-*invariant*). This means that for all $X_1, \dots, X_k \in \mathfrak{g}$ and $g \in G$, we have

$$\varphi(\mathrm{ad}_g(X_1), \dots, \mathrm{ad}_g(X_k)) = \varphi(X_1, \dots, X_k).$$

Henceforth, we will assume G is a matrix Lie group for simplicity. Thus $\mathrm{ad}_g(X) = gXg^{-1}$.

By the polarisation formula for symmetric multilinear maps, every element $\varphi \in I^k(G)$ is completely determined its polarisation $P_\varphi : \mathfrak{g} \to \mathbb{K}$, given by $P_\varphi(X) = \varphi(X, \dots, X)$. Note that P_φ is

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then ad-invariant and homogeneous of degree k, that is

$$P_\varphi(\lambda X) = \lambda^k P_\varphi(X) \qquad \forall \lambda \in \mathbb{K}.$$

Conversely, if a function $P : \mathfrak{g} \to \mathbb{K}$ is ad-invariant and homogeneous of degree k, then P defines a unique element $\varphi \in I^k(G)$ by setting $\varphi(X, \ldots, X) = P(X)$, and extending to the whole domain by the polarisation formula. Therefore, we can identify $I^k(G)$ with the space of ad-invariant maps $\mathfrak{g} \to \mathbb{K}$ which are homogeneous of degree k. In either interpretation, $I^k(G)$ is a $\mathbb{K}$-vector space. Even more, the space

$$I^\bullet(G) = \bigoplus_{k \geq 0} I^k(G)$$

is naturally a $\mathbb{K}$-algebra. We refer to elements of $I^\bullet(G)$ as *invariant polynomials*.[1]

1.2 The Chern–Weil homomorphism

In this subsection, P is a principal G-bundle over a (smooth) manifold M. We will assume P is defined by a trivialising open cover $\{U_\alpha\}$ of M and a collection of transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$. Also assume $\mathbb{K} = \mathbb{R}$ for simplicity; all of the results below still hold when $\mathbb{K} = \mathbb{C}$.

The Chern–Weil construction, which we will detail shortly, assigns to every invariant polynomial in $I^k(G)$ a cohomology class in $H^{2k}(M)$ using the structure of P. These cohomology classes are called *characteristic classes*, and are invariants of P. If two principal bundles yield different characteristic classes, they are not isomorphic. This provides a partial answer to our opening question, but only for *principal* bundles. The case for *vector* bundles will be explained in due course.

Lemma 1.1. *Let $\varphi \in I^k(G)$ be an invariant polynomial. Let $A = \{A_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})\}$ be a connection on P with curvature $F = \{F_\alpha \in \Omega^2(U_\alpha; \mathfrak{g})\}$. Then $\varphi(F_\alpha, \ldots, F_\alpha) \in \Omega^{2k}(U_\alpha)$, and these local $2k$-forms patch together to give a well-defined global $2k$-form $\varphi(F, \ldots, F) \in \Omega^{2k}(M)$.*

Proof. Since φ and F_α are both multilinear, it follows that $\varphi(F_\alpha, \ldots, F_\alpha)$ is multilinear in its $2k$ entries. Since φ is symmetric and F_α is skew-symmetric, we see that $\varphi(F_\alpha, \ldots, F_\alpha)$ is alternating. Thus $\varphi(F_\alpha, \ldots, F_\alpha) \in \Omega^{2k}(U_\alpha)$, which is the first claim.

Recalling that $F \in \Omega^2(M; \mathrm{ad}P)$, this means that $F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}$ on $U_\alpha \cap U_\beta$. Hence

$$\varphi(F_\alpha, \ldots, F_\alpha) = \varphi(g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}, \ldots, g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}) = \varphi(F_\beta, \ldots, F_\beta),$$

where the ad-invariance of φ was used. So $\varphi(F, \ldots, F)$ is a well-defined global $2k$-form on M. □

The next theorem is the essence of Chern–Weil theory. We write $\mathcal{A}(P)$ for the space of connections on P. Recall that this is an affine space modelled on $\Omega^1(M; \mathrm{ad}P)$.

Theorem 1.2 (Chern–Weil theorem)**.** *Let $\varphi \in I^k(G)$ be an invariant polynomial, and let $A, A' \in \mathcal{A}(P)$ be two connections with respective curvatures F^A, $F^{A'} \in \Omega^2(M; \mathrm{ad}P)$. Then:*

(a) $\varphi(F^A, \ldots, F^A)$ is closed in $\Omega^{2k}(M)$. Thus it represents a cohomology class in $H^{2k}(M)$.

(b) $\varphi(F^{A'}, \ldots, F^{A'}) = \varphi(F^A, \ldots, F^A)$ in $H^{2k}(M)$.

[1] Lemma 3.1 shows how these elements are really just polynomials in the traditional sense.

Proof. It suffices to work with the local curvature forms F_α^A, $F_\alpha^{A'}$ over an open subset $U_\alpha \subset M$. To prove (a), we need two identities. The first one is the Bianchi identity $dF_\alpha^A + [A_\alpha, F_\alpha^A] = 0$. The second identity is that for all $X, X_1, \ldots, X_k \in \mathfrak{g}$ we have

$$\varphi([X, X_1], X_2, \ldots, X_k) + \ldots + \varphi(X_1, \ldots, X_{k-1}, [X, X_k]) = 0.$$

This follows from the fact that $\varphi(e^{tX} X_1 e^{-tX}, \ldots, e^{tX} X_k e^{-tX})$ is constant with respect to t by the ad-invariance of φ, and thus

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \varphi(e^{tX} X_1 e^{-tX}, \ldots, e^{tX} X_k e^{-tX}) \\ &= \varphi(XX_1 - X_1X, X_2, \ldots, X_k) + \ldots + \varphi(X_1, \ldots, X_{k-1}, XX_k - X_kX) \\ &= \varphi([X, X_1], X_2, \ldots, X_k) + \ldots + \varphi(X_1, \ldots, X_{k-1}, [X, X_k]). \end{aligned}$$

Using these two identities, we have (using also the Leibniz rule and the multilinearity of φ)

$$\begin{aligned} d\varphi(F_\alpha^A, \ldots, F_\alpha^A) &= \varphi(dF_\alpha^A, F_\alpha^A, \ldots, F_\alpha^A) + \ldots + \varphi(F_\alpha^A, \ldots, F_\alpha^A, dF_\alpha^A) \\ &= -\varphi([A_\alpha, F_\alpha^A], F_\alpha^A, \ldots, F_\alpha^A) - \ldots - \varphi(F_\alpha^A, \ldots, F_\alpha^A, [A_\alpha, F_\alpha^A]) = 0, \end{aligned}$$

which is (a). Now we prove (b). We need to show that

$$\varphi(F^{A'}, \ldots, F^{A'}) - \varphi(F^A, \ldots, F^A) \tag{1}$$

is exact. To begin, we need another identity which reads as follows. If $F_1, \ldots, F_{k-1}$ are even-degree forms on U_α, then

$$\sum_{i=1}^k \varphi(F_1, \ldots, F_{i-1}, [A_\alpha, F_i], F_{i+1}, \ldots, F_k) = 0. \tag{2}$$

(See [Nic07, p318] for the routine derivation.) Now we set $C_\alpha = A'_\alpha - A_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})$, so that for each $t \in [0,1]$ we get another local connection 1-form $A_\alpha^t = A_\alpha + tC_\alpha$. Let $F_\alpha^t \in \Omega^2(U_\alpha; \mathfrak{g})$ be its curvature. We have

$$F_\alpha^t = dA_\alpha^t + A_\alpha^t \wedge A_\alpha^t = F_\alpha^t + t(dC_\alpha + [A_\alpha, C_\alpha]) + \frac{t^2}{2}[C_\alpha, C_\alpha],$$

so that

$$\frac{d}{dt} F_\alpha^t = dC_\alpha + [A_\alpha, C_\alpha] + t[C_\alpha, C_\alpha] = dC_\alpha + [A_\alpha^t, C_\alpha]. \tag{3}$$

We may now express the quantity (1) as

$$\varphi(F^{A'}, \ldots, F^{A'}) - \varphi(F^A, \ldots, F^A) = \int_0^1 \frac{d}{dt} \varphi(F_\alpha^t, \ldots, F_\alpha^t)\, dt = k \int_0^1 \varphi(F_\alpha^t, \ldots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]), \tag{4}$$

where we have used (3), the Leibniz rule and the symmetry of φ. We claim the integrand is exact, in particular equal to $d\varphi(F_\alpha^t, \ldots, F_\alpha^t, C_\alpha)$. Indeed, by the Bianchi identity $dF_\alpha^t = -[A_\alpha^t, F_\alpha^t]$ and (2), we have

$$\begin{aligned} d\varphi(F_\alpha^t, \ldots, F_\alpha^t, C_\alpha) &= \varphi(dF_\alpha^t, F_\alpha^t, \ldots, F_\alpha^t, C_\alpha) + \ldots + \varphi(F_\alpha^t, \ldots, F_\alpha^t, dF_\alpha^t, C_\alpha) \\ &\quad + \varphi(F_\alpha^t, \ldots, F_\alpha^t, dC_\alpha) \\ &= -\varphi([A_\alpha^t, F_\alpha^t], F_\alpha^t, \ldots, F_\alpha^t, C_\alpha) - \ldots - \varphi(F_\alpha^t, \ldots, F_\alpha^t, [A_\alpha^t, F_\alpha^t], C_\alpha) \\ &\quad + \varphi(F_\alpha^t, \ldots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]) - \varphi(F_\alpha^t, \ldots, F_\alpha^t, [A_\alpha^t, C_\alpha]) \\ &= \varphi(F_\alpha^t, \ldots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]). \end{aligned}$$

Putting this back into (4) gives

$$\varphi(F^{A'}, \dots, F^{A'}) - \varphi(F^A, \dots, F^A) = d\int_0^1 k\varphi(F^t_\alpha, \dots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha]),$$

so the difference is exact, as required. □

Theorem 1.2 implies that given a principal G-bundle P over M, and $k \in \mathbb{N}$, there is a map

$$I^k(G) \to H^{2k}(M) \qquad \varphi \mapsto \varphi(F^A, \dots, F^A), \tag{5}$$

where A is *any* connection on P. It therefore makes sense to write $\varphi(P)$ in place of $\varphi(F^A, \dots, F^A)$. The map (5) is called the *Chern–Weil map*. This induces a map

$$\mathrm{cw}_P : I^\bullet(G) \to H^{\mathrm{even}}(M) := \bigoplus_{k\geq 0} H^{2k}(M), \qquad \varphi \mapsto \varphi(P),$$

which is in fact an algebra homomorphism (we will not bother checking this). For this reason we often refer to the Chern–Weil map as the *Chern–Weil homomorphism*. Elements in the image of cw_P are called the *characteristic classes* of P.

Corollary 1.3. *The Chern–Weil homomorphism has the following properties.*

(a) If P is trivial, then $\ker \mathrm{cw}_P = I^\bullet(G)$. In other words, all characteristic classes of P vanish.

(b) If $f : M \to N$ is a morphism of smooth manifolds, and P is a principal G-bundle over M, then $\mathrm{cw}_{F^*P} = F^*\mathrm{cw}_P$.

Proof. If P is trivial, then we can pick a flat connection A, meaning that $F^A \equiv 0$. For every $\varphi \in I^\bullet(G)$, the multilinearity of φ gives $\mathrm{cw}_P(\varphi) = \varphi(P) = \varphi(F^A, \dots, F^A) \equiv 0$, so cw_P has no image. We leave the proof of (b) to the reader. □

By (b) above, isomorphic principal bundles over a given smooth manifold have identical characteristic classes (note: the converse is false in general). Thus, characteristic classes give us a way of telling whether two principal bundles are non-isomorphic.

However, our original goal was to find a way to distinguish *vector* bundles, so we need to transplant all these notions to vector bundles. For a complex vector bundle E, the procedure is as follows: choose a hermitian metric h on E, then form the unitary frame bundle $\mathcal{F}_h(E)$ with respect to h. This is a principal $U(r)$-bundle. The characteristic classes of E are defined to be the characteristic classes of $\mathcal{F}_h(E)$, which are given by the Chern–Weil construction above. This is well-defined, because if h' is another hermitian metric on E, then $\mathcal{F}_{h'}(E) \cong \mathcal{F}_h(E)$ (one constructs this isomorphism using Gram–Schmidt).

Equivalently, we can choose an affine connection ∇ on E compatible with some (arbitrary) hermitian metric. Compatibility implies that its curvature F_∇ is a $\mathfrak{u}(r)$-valued 2-form on M, so for each $\varphi \in I^k(U(r))$ we have $\varphi(F_\nabla, \dots, F_\nabla) \in \Omega^{2k}(M)$. This is equivalent to the above method, since choosing such a ∇ is equivalent to choosing a hermitian metric h on E and a principal connection on $\mathcal{F}_h(E)$. Thus, $\varphi(F_\nabla, \dots, F_\nabla)$ is independent of the choice of ∇, as long as ∇ is compatible with some hermitian metric. We get a well-defined map

$$\mathrm{cw}_E : I^\bullet(U(r)) \to H^{\mathrm{even}}(M), \qquad \varphi \mapsto \varphi(E) := \varphi(F_\nabla, \dots, F_\nabla). \tag{6}$$

The image of cw_E defines the characteristic classes of E. From this discussion, we see that Corollary 1.3 generalises to the vector bundle E.

Remark. If E is a real vector bundle, we can define characteristic classes of E by repeating the above construction, but using a real metric g on E, and a connection on the orthonormal frame bundle $\mathcal{F}_g(E)$. Equivalently, use an affine connection on E compatible with some metric. If E is an oriented real vector bundle, look at the oriented orthonormal frame bundle $\mathcal{F}_g^S(E)$ instead. Note: this means 'the characteristic classes of E' depend on whether it is seen as a real or complex vector bundle, and with or without orientation in the real case.

2 Chern classes

To see the Chern–Weil construction in action, we will introduce the highly important *Chern classes* for a complex vector bundle. We will then compute the Chern classes of the tangent bundle of complex projective space, $\mathbb{CP}^n$.

2.1 Defining Chern classes

Let $r \in \mathbb{N}$. For $X \in \mathfrak{u}(r)$, consider the *Chern polynomial*

$$c_t(X) = \det\left(\mathbb{1} - \frac{t}{2\pi i}X\right),$$

which is a polynomial in t whose coefficients are functions of the entries of X. Since $A \mapsto \det(\mathbb{1}+A)$ is invariant under conjugation in $GL(r, \mathbb{C})$, we have in particular that $c_t(gXg^{-1}) = c_t(X)$ for $g \in U(r)$. This implies that if we define $c_1, c_2, \dots$ by

$$c_t(X) = \sum_{k\geq 0} c_k(X)t^k, \tag{7}$$

then $c_k(gXg^{-1}) = c_k(X)$ for each k and each $g \in U(r)$. Moreover, $c_k(X)$ is a function of k entries of X. Thus $c_k(tX) = t^k c_k(X)$. Since $c_k(X) \in \mathbb{R}$ (see below), all this implies $c_k \in I^k(U(r))$. The Chern classes of a complex vector bundle are obtained by applying the Chern–Weil construction to c_k:

Definition 2.1. Let E be a complex vector bundle of rank r. The *Chern classes* of E are the cohomology classes $c_k(E) = \mathrm{cw}_E(c_k) \in H^{2k}(M)$, where cw_E was defined in (6).

We will now find an explicit expression for $c_k(E)$. Let us first consider $c_k(X)$ where $X \in \mathfrak{u}(r)$. Thus X is a skew-symmetric and hermitian $r \times r$ matrix. We know from linear algebra that X is unitarily diagonalisable, say by $T \in U(r)$. Moreover, X has purely imaginary eigenvalues so that

$$TXT^{-1} = \mathrm{diag}(\lambda_1, \dots, \lambda_r), \qquad \lambda_j \in i\mathbb{R}.$$

Writing $\sigma_k(x_1, \dots, x_r)$ for the k-th elementary symmetric function in $x_1, \dots, x_r$, it follows that

$$\begin{aligned} c_t(X) = c_t(TXT^{-1}) &= \det\left(\mathrm{diag}\left(1 - \frac{\lambda_1 t}{2\pi i}, \dots, 1 - \frac{\lambda_r t}{2\pi i}\right)\right) = \left(1 - \frac{\lambda_1 t}{2\pi i}\right)\cdots\left(1 - \frac{\lambda_r t}{2\pi i}\right) \\ &= \sum_{k=0}^{r} \sigma_k\left(-\frac{\lambda_1 t}{2\pi i}, \dots, -\frac{\lambda_r t}{2\pi i}\right) = \sum_{k=0}^{r}\left(-\frac{1}{2\pi i}\right)^k \sigma_k(\lambda_1, \dots, \lambda_r)t^k. \end{aligned}$$

By (7), we have $c_k(X) = (-\frac{1}{2\pi i})^k \sigma_k(\lambda_1, \ldots, \lambda_r)$ where the $\lambda_j \in i\mathbb{R}$ are the eigenvalues of X. Writing $\lambda_j = i\hat{\lambda}_j$ where $\hat{\lambda}_j \in \mathbb{R}$, we get $c_k(X) = (-\frac{1}{2\pi})^k \sigma_k(\hat{\lambda}_1, \ldots, \hat{\lambda}_k) \in \mathbb{R}$ as claimed above. Going back to our vector bundle E, this means

$$c_k(E) = \left(-\frac{1}{2\pi i}\right)^k \sigma_k(\lambda_1(F_\nabla), \ldots, \lambda_r(F_\nabla)) \in H^{2k}(M), \tag{8}$$

where ∇ is a connection on E compatible with some hermitian metric, and the $\lambda_j(F_\nabla)$ are the eigenvalues of its curvature F_∇, which are purely imaginary. In particular,

$$c_1(E) = -\frac{1}{2\pi i} \operatorname{tr}(F_\nabla), \qquad c_k(E) = 0 \text{ for } k > r. \tag{9}$$

Of course, the Chern classes $c_k(E)$, $k \leq r$ may also vanish due to dimensionality concerns of the base manifold M or otherwise (e.g. if E is trivial).

It is often convenient to collect all information about the Chern classes into a single expression. One way to do this is to look at the *Chern polynomial* $c_t(E) = \sum_{k=0}^r c_k(E)t^k$. However, knowing $c_t(E)$ is equivalent to knowing the *total Chern class* of E:

Definition 2.2. The *total Chern class* of a rank r complex vector bundle E is

$$c(E) = 1 + c_1(E) + c_2(E) + \ldots + c_r(E) \in H^{\text{even}}(E).$$

Namely, $c_k(E)$ is the component of $c(E)$ in $H^{2k}(E)$.

2.2 Chern classes of $T\mathbb{CP}^n$

In this subsection, we compute the Chern classes of the tangent bundle of $\mathbb{CP}^n$, following [MS74, §14]. The end result is as follows:

Theorem 2.3. *The total Chern class of* $T\mathbb{CP}^n$ *is* $c(T\mathbb{CP}^n) = (1-\alpha)^{n+1}$*, where* $\alpha \neq 0$ *is a generator for* $H^2(\mathbb{CP}^n)$*. In fact* $\alpha = c_1(\gamma^1)$*, the first Chern class of the tautological line bundle over* $\mathbb{CP}^n$*.*

The product of a with itself is understood as the wedge product. Thus:

Corollary 2.4. *The Chern classes of* $T\mathbb{CP}^n$ *are* $c_k(T\mathbb{CP}^n) = (-1)^k \binom{n+1}{k} \alpha^{\wedge k}$*, for* $k = 1, \ldots, n$*.*

Since $\alpha \neq 0$, we conclude from Corollary 1.3 (rather, the version for vector bundles) that:

Corollary 2.5. $T\mathbb{CP}^n$ *is nontrivial.*

To prove Theorem 2.3, we start with two simple lemmas concerning the arithmetic of Chern classes. The first lemma expresses the Chern polynomial of a Whitney sum of vector bundles in terms of the constitutent Chern polynomials:

Lemma 2.6. *If* E *and* E' *are two complex vector bundles over* M*, then*

$$c_t(E \oplus E') = c_t(E)c_t(E').$$

Proof. Suppose E has rank r and E' has rank r'. Let h and h' be hermitian metrics on E and E' respectively. Let ∇^E be a connection on E compatible with h, and $\nabla^{E'}$ a connection on E' compatible with h'. Compatibility means their connection 1-forms A, A' locally take values in $\mathfrak{u}(r)$ and $\mathfrak{u}(r')$ respectively. Then the block-diagonal 1-forms $A \oplus A'$ define a connection $\nabla^{E\oplus E'}$ on $E \oplus E'$, which is compatible with the hermitian metric $h \oplus h'$. Moreover, $A \oplus A'$ locally takes values in $\mathfrak{u}(r) \oplus \mathfrak{u}(r') \subset \mathfrak{u}(r + r')$. The curvature 2-forms are related by

$$F_{\nabla^{E\oplus E'}} = F_{\nabla^E} \oplus F_{\nabla^{E'}}.$$

Thus,

$$\begin{aligned} c_t(E \oplus E') &= \det\left(\mathbb{1}_{r+r'} - \frac{t}{2\pi i} F_{\nabla^{E\oplus E'}}\right) \\ &= \det\left(\mathbb{1}_r - \frac{t}{2\pi i} F_{\nabla^E}\right) \det\left(\mathbb{1}_{r'} - \frac{t}{2\pi i} F_{\nabla^{E'}}\right) = c_t(E) c_t(E'). \end{aligned}$$

□

A consequence of the above lemma is that $c(E \oplus E') = c(E)c(E')$, since the total Chern class is just the Chern polynomial evaluated at $t = 1$.

Lemma 2.7. *If E is a complex vector bundle and $\bar{E}$ is its conjugate bundle, then for each $k \in \mathbb{N}$ we have*

$$c_k(\bar{E}) = (-1)^k c_k(E).$$

Proof. Let $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ be a connection on E compatible with a hermitian metric h. Then

$$\overline{\nabla} : \Gamma(TM) \times \Gamma(\bar{E}) \to \Gamma(\bar{E}), \qquad \overline{\nabla}_X s = \overline{\nabla_X s},$$

defines a connection on $\bar{E}$ compatible with the hermitian metric $\bar{h}$ on $\bar{E}$. If A is the connection 1-form of ∇, defined locally by

$$\nabla_X \{e_1^\alpha, \dots, e_r^\alpha\} = \{e_1^\alpha, \dots, e_r^\alpha\} A_\alpha(X)$$

where $\{e_1^\alpha, \dots, e_r^\alpha\}$ is a local frame for E over an open set $U_\alpha \subset M$, then we see that $\overline{\nabla}$ has connection 1-form $\overline{A}$. Using the formula $F_{\overline{\nabla}} = d\overline{A} + \overline{A} \wedge \overline{A}$, we see that the curvature forms are related by $F_{\overline{\nabla}} = \overline{F_\nabla}$. Then by (8) and the fact that F_∇ has purely imaginary eigenvalues (by virtue of taking values in $\mathfrak{u}(r)$), we have

$$\begin{aligned} c_k(\bar{E}) &= \left(-\frac{1}{2\pi i}\right)^k \sigma_k(\lambda_1(F_{\overline{\nabla}}), \dots, \lambda_r(F_{\overline{\nabla}})) = \left(-\frac{1}{2\pi i}\right)^k \sigma_k(\overline{\lambda_1(F_\nabla)}, \dots, \overline{\lambda_r(F_\nabla)}) \\ &= \left(-\frac{1}{2\pi i}\right)^k \sigma_k(-\lambda_1(F_\nabla), \dots, -\lambda_r(F_\nabla)) = (-1)^k c_k(E), \end{aligned}$$

as claimed. □

Corollary 2.8. *If E is a complex vector bundle and E^* is its dual bundle, then for each $k \in \mathbb{N}$ we have*

$$c_k(E^*) = (-1)^k c_k(E).$$

Proof. Let h be a hermitian metric on E. For each $x \in M$, define a map $f_x : \bar{E}_x \to E_x^*$ by $f_x(v) = h_x(v, \cdot)$. It is routine to check that this induces an isomorphism $\bar{E} \cong E^*$. The corollary follows from Lemma 2.7. □

Proof of Theorem 2.3. Let γ^1 be the tautological line bundle over $\mathbb{CP}^n$, so that the fibre over $\ell \in \mathbb{CP}^n$ is $\ell \subset \mathbb{C}^{n+1}$. Let ω^n be the orthogonal complement of γ^1 in $\mathbb{C}^{n+1}$. To be specific, for each $\ell \in \mathbb{CP}^n$, we define $\omega^n_\ell = (\gamma^1_\ell)^\perp = \ell^\perp$, where the orthocomplement is taken in $\mathbb{C}^{n+1}$ using the standard hermitian inner product. Then $\omega^n \oplus \gamma^1 = \varepsilon^{n+1}$, the trivial complex bundle over $\mathbb{CP}^n$.

We will first show that $T\mathbb{CP}^n \cong \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \omega^n)$. Let $\ell \in \mathbb{CP}^n$. Observe that the graph of any complex linear map $f \in \mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp)$ is a line in $\mathbb{C}^{n+1}$ close to ℓ. So we can identify a neighbourhood of ℓ in $\mathbb{CP}^n$ with a neighbourhood of zero in $\mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp)$. Taking tangent spaces, we get that $T_\ell\mathbb{CP}^n \cong \mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp)$. (Here we used that $\mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp)$ is a $\mathbb{C}$-vector space, so its tangent spaces are itself.) As $\ell \in \mathbb{CP}^n$ was arbitrary, this induces a bundle isomorphism $T\mathbb{CP}^n \cong \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \omega^n)$ as was claimed.

Adding the trivial line bundle ε^1 over $\mathbb{CP}^n$ to both sides of this isomorphism, and using the fact that $\varepsilon^1 \cong (\gamma^1)^* \otimes \gamma^1 \cong \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \gamma^1)$, we get

$$T\mathbb{CP}^n \oplus \varepsilon^1 \cong \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \omega^n \oplus \gamma^1) = \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \varepsilon^{n+1}) \cong \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \mathbb{C}^{\oplus(n+1)}) \cong [(\gamma^1)^*]^{\oplus(n+1)}. \quad (10)$$

Since ε^1 is trivial, its total Chern class is 1 (as $c_1(\varepsilon^1)$ vanishes by Corollary 1.3). Now using the above with Lemma 2.7 and Corollary 2.8, we have

$$c(T\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \varepsilon^1) = c([(\gamma^1)^*]^{\oplus(n+1)}) = c((\gamma^1)^*)^{n+1} = (1 - c_1(\gamma^1))^{n+1}.$$

It remains to show that $c_1(\gamma^1) \neq 0$. To see this, we need to choose a connection on γ^1 compatible with some hermitian metric. There is a natural hermitian metric on γ^1 given by restricting the canonical one on ε^{n+1}. Let ∇ be the Chern connection associated to this metric.[2] A computation (see [Liu18, Example 1.2.12]) yields that the curvature of ∇ is locally

$$F_\nabla = -\sum_{k,\ell=1}^{n} \frac{(1+|\theta|^2)\delta_{k\ell} - \bar{\theta}_k\theta_\ell}{(1+|\theta|^2)^2} d\theta_k \wedge d\bar{\theta}_\ell,$$

where $\theta = (\theta_1, \dots, \theta_n)$ are local coordinates for $\mathbb{CP}^n$, defined in the standard way. This does not vanish identically, so by (9)

$$c_1(\gamma^1) = -\frac{1}{2\pi i}\,\mathrm{tr}(F_\nabla) = -\frac{1}{2\pi i} F_\nabla \neq 0 \quad \text{in } H^2(\mathbb{CP}^n). \quad (11)$$

Here we have used that F_∇ takes values in $\mathbb{C}$ to get rid of the trace. Finally, by Lemma 2.9 below, $c_1(\gamma^1)$ generates $H^2(\mathbb{CP}^n)$. □

The next lemma was used just now, and will be needed again when we sketch the proof of the Hirzebruch signature theorem (Theorem 3.9).

[2] The Chern connection of a holomorphic, hermitian vector bundle (E, h) over a complex manifold is the unique connection on E which is compatible with h and is holomorphic (in a suitably defined way). Here γ^1 inherits a holomorphic structure from ε^{n+1}. There is an explicit formula for the Chern connection in terms of the metric, as well as its curvature. (Compare with the Levi-Civita connection on a Riemannian manifold and the Koszul formula.)

Lemma 2.9. *For all $n, k \in \mathbb{N}$ with $2 \leq k \leq 2n$ and k even, the space $H^k(\mathbb{CP}^n)$ is a one-dimensional real vector space generated by $c_1(\gamma^1)^{k/2}$, where γ^1 is the tautological bundle over $\mathbb{CP}^n$. Moreover, $c_1(\gamma^1)$ is everywhere nonzero and*

$$\int_{\mathbb{CP}^n} c_1(\gamma^1)^n = 1. \tag{12}$$

Proof. See [Dup03, Example A15] for a proof of the first part which is consistent with our definition of Chern class (i.e. via the Chern–Weil construction). There it is also shown that $c_1(\gamma^1)$ is everywhere nonzero. See [Ath18, §2.7] for a proof of (12) when $n = 1$ (beware of sign conventions). ☐

2.3 Some topological consequences

For us to conclude that $T\mathbb{CP}^n$ is nontrivial, we only needed one of its Chern classes to be nonzero. This makes the above computations seem like an overkill since we computed *all* the Chern classes of $T\mathbb{CP}^n$. The topological results in this subsection should convince the reader that we have not done this work in vain. The material here is based on [Ath18, §2.8].

In this subsection, we take care to distinguish real vector bundles from complex ones. If E is a complex vector bundle of rank r, then its realification $E_\mathbb{R}$ is a real vector bundle of rank $2r$. The transition functions of $E_\mathbb{R}$ are those of E under the image of $GL(r, \mathbb{C})$ in $GL(2r, \mathbb{R})$. On the contrary, a real vector bundle ξ complexifies to a complex vector bundle $\xi_\mathbb{C} = \xi \otimes_\mathbb{R} \mathbb{C}$ of equal rank. We write $T\mathbb{CP}^n$ for the *complex* tangent bundle of $\mathbb{CP}^n$. We need one simple fact:

Lemma 2.10. *If E is a complex vector bundle, then $(E_\mathbb{R})_\mathbb{C} \cong E \oplus E^*$ as complex vector bundles.*

Proof. If V is a complex vector space, then the map

$$V \otimes_\mathbb{R} \mathbb{C} \to V \oplus \overline{V}, \qquad v \otimes_\mathbb{R} z \mapsto (zv, \bar{z}v)$$

is an isomorphism of complex vector spaces. Applying this to the fibres of E, we get $(E_\mathbb{R})_\mathbb{C} \cong E \oplus \bar{E}$. But $\bar{E} \cong E^*$ by the argument of Corollary 2.8, so $(E_\mathbb{R})_\mathbb{C} \cong E \oplus E^*$. ☐

The next two theorems are topological consequences of the computations in Section 2.2.

Theorem 2.11. *There is no compact smooth $(4n+1)$-dimensional manifold with boundary $\mathbb{CP}^{2n}$.*

Proof. For a contradiction, suppose M is a $(4n+1)$-dimensional smooth manifold with $\partial M = \mathbb{CP}^{2n}$. Let $\iota : \mathbb{CP}^{2n} \hookrightarrow M$ be the inclusion. By the collar neighbourhood theorem, we have

$$T\partial M \oplus \varepsilon^1_\mathbb{R} \cong \iota^*(TM)$$

where $\varepsilon^1_\mathbb{R}$ is a trivial rank 1 real vector bundle over $\mathbb{CP}^{2n}$. Complexifying, we get

$$((T\mathbb{CP}^{2n})_\mathbb{R})_\mathbb{C} \oplus \varepsilon^1_\mathbb{C} \cong (\iota^*(TM))_\mathbb{C} \cong \iota^*(TM_\mathbb{C}) \tag{13}$$

as complex vector bundles. Now using Lemma 2.10 and then (10), we have

$$\begin{aligned} ((T\mathbb{CP}^{2n})_\mathbb{R})_\mathbb{C} \oplus \varepsilon^2_\mathbb{C} &\cong (T\mathbb{CP}^{2n} \oplus T^*\mathbb{CP}^{2n}) \oplus (\varepsilon^1_\mathbb{C} \oplus \varepsilon^1_\mathbb{C}) \\ &\cong (T\mathbb{CP}^{2n} \oplus \varepsilon^1_\mathbb{C}) \oplus (T^*\mathbb{CP}^{2n} \oplus (\varepsilon^1_\mathbb{C})^*) \\ &\cong [(\gamma^1)^*]^{\oplus(2n+1)} \oplus (\gamma^1)^{\oplus(2n+1)}, \end{aligned}$$

where γ^1 is the tautological line bundle over $\mathbb{CP}^{2n}$. Lemma 2.6 now gives that

$$
\begin{aligned}
c(((T\mathbb{CP}^{2n})_{\mathbb{R}})_{\mathbb{C}}) &= (1 - c_1(\gamma^1))^{2n+1}(1 + c_1(\gamma^1))^{2n+1} = (1 - c_1(\gamma^1)^2)^{2n+1} \\
&= \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} c_1(\gamma^1)^{2k}.
\end{aligned} \tag{14}
$$

Let $\omega \in \Omega^{4n}(M)$ be a closed form representing the cohomology class $c_{2n}(TM_{\mathbb{C}})$. Then by Corollary 1.3, followed by (13), Lemma 2.6 and then (14), the cohomology class of $\iota^*\omega$ satisfies

$$
[\iota^*\omega] = \iota^* c_{2n}(TM_{\mathbb{C}}) = c_{2n}(\iota^*(TM_{\mathbb{C}})) = c_{2n}(((T\mathbb{CP}^{2n})_{\mathbb{R}})_{\mathbb{C}}) = (-1)^n \binom{2n+1}{n} c_1(\gamma^1)^{2n} \neq 0.
$$

This means $\iota^*\omega$ is a top form on $\mathbb{CP}^{2n}$, so by Lemma 2.9 it is everywhere nonvanishing (hence cannot change sign). Using this, the fact that $d\omega = 0$ and Stokes' theorem, we get

$$
0 = \int_M d\omega = \int_{\partial M} \iota^*\omega \neq 0.
$$

This yields the desired contradiction. □

Theorem 2.12. *There is no smooth embedding of $\mathbb{CP}^4$ into $\mathbb{R}^{11}$.*

Proof. Suppose this were false, so $\iota : \mathbb{CP}^4 \hookrightarrow \mathbb{R}^{11}$ is an embedding. Then there is a subbundle ξ of $\varepsilon^{11} := \iota^* T\mathbb{R}^{11}$ which is the normal bundle with respect to this embedding, meaning

$$
(T\mathbb{CP}^4)_{\mathbb{R}} \oplus \xi \cong \varepsilon^{11}. \tag{15}
$$

Complexifying both sides, and then using Corollary 1.3 and Lemma 2.6, we get

$$
c(((T\mathbb{CP}^4)_{\mathbb{R}})_{\mathbb{C}}) c(\xi_{\mathbb{C}}) = 1.
$$

Writing $\alpha = c_1(\gamma^1)$ for the first Chern class of the tautological bundle over $\mathbb{CP}^4$, and using (14), this gives

$$
c(\xi_{\mathbb{C}}) = \frac{1}{(1 - X^2)^5} = \sum_{k \geq 0} (-1)^k \binom{-5}{k} X^{2k} = 1 + 5X^2 + 15X^4.
$$

The last equality is because $X^6, X^8, \ldots$ all vanish on $\mathbb{CP}^4$ which has real dimension 8. By the definition of total Chern class, this shows that $c_4(\xi_{\mathbb{C}}) = 15X^4$. But Lemma 2.9 says that $15X^4$ is nonzero in $H^8(\mathbb{CP}^4)$, so (9) gives that $\operatorname{rank}(\xi_{\mathbb{C}}) \geq 2$ as a complex vector bundle. Hence ξ has real rank ≥ 4. Since ranks are additive under Whitney sums, $(T\mathbb{CP}^4)_{\mathbb{R}} \oplus \xi$ has real rank at least $8 + 4 = 12$. This contradicts (15) since ε^{11} has rank 11. □

3 Generating characteristic classes, and a glimpse of index theory

In this section, we will describe a way of generating characteristic classes from power series in one variable, which builds on top of the Chern–Weil construction. This gives rise to several important characteristic classes beyond the Chern classes that are used in profound index theorems connecting algebraic geometry, algebraic topology and differential geometry. We will only scratch the very surface of this topic. The treatment in Sections 3.1 and 3.2 is influenced by [Nic07, §8.2].[3]

[3] In hindsight, a neater (and more common) approach would use *multiplicative sequences*; see [MS74, §19]).

3.1 Characteristic classes from power series: complex vector bundles

Let E be a complex vector bundle of rank r over M. The Chern–Weil map $\mathrm{cw}_E : I^\bullet(U(r)) \to H^{\mathrm{even}}(M)$ defines the characteristic classes of E. We will first explicitly characterise $I^\bullet(U(r))$ to understand what characteristic classes can arise.

Lemma 3.1. *For each $k \in \mathbb{N}$, there is a bijective correspondence between $I^k(U(r))$ and $\mathbb{R}^k_{\mathrm{sym}}[x_1, \dots, x_r]$, the space of symmetric polynomials of homogeneous degree k in $x_1, \dots, x_r$ with real coefficients. The correspondence is given by*

$$\begin{aligned} I^k(U(r)) &\longleftrightarrow \mathbb{R}^k_{\mathrm{sym}}[x_1, \dots, x_r] \\ \varphi &\mapsto \qquad\quad P_\varphi \\ \varphi_P &\leftarrow\!\shortmid \qquad\quad P, \end{aligned}$$

where

$$P_\varphi(x_1, \dots, x_r) = \varphi\left(\begin{bmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_r \end{bmatrix}\right)$$

and

$$\varphi_P(X) = P(\hat{\lambda}_1(X), \dots, \hat{\lambda}_r(X)),$$

where $i\hat{\lambda}_1(X), \dots, i\hat{\lambda}_r(X)$ are the eigenvalues of $X \in \mathfrak{u}(r)$.

We will skip the proof which is rudimentary. (When proving this, it is crucial to use the ad-invariance of $\varphi \in I^k(U(r))$ and the fact that $X \in \mathfrak{u}(r)$ has purely imaginary eigenvalues.) Now considering all k at once, the following corollary is immediate.

Corollary 3.2. *There is a bijective correspondence between $I^\bullet(U(r))$ and $\mathbb{R}_{\mathrm{sym}}[[x_1, \dots, x_r]]$, the ring of symmetric formal power series in $x_1, \dots, x_r$ with real coefficients. The correspondence is given as in Lemma 3.1 for the degree k part, $k \in \mathbb{N}$.*

Corollary 3.2 is in fact a special case of the classic *Chevalley restriction theorem*.

We now describe how a symmetric power series $g \in \mathbb{R}_{\mathrm{sym}}[[x_1, \dots, x_r]]$ can be used to obtain a characteristic class for E. By the fundamental theorem of symmetric polynomials, g can be written

$$g(x_1, \dots, x_r) = p_g(\sigma_1(x_1, \dots, x_r), \dots, \sigma_r(x_1, \dots, x_r))$$

where σ_j is the j-th elementary symmetric function, and p_g is a formal power series. Via the correspondence of Corollary 3.2, g gets mapped to $\varphi_g \in I^\bullet(U(r))$ which is given by

$$\begin{aligned} \varphi_g(X) &= g(\hat{\lambda}_1(X), \dots, \hat{\lambda}_r(X)) \\ &= p_g(\sigma_1(\hat{\lambda}_1(X), \dots, \hat{\lambda}_r(X)), \dots, \sigma_r(\hat{\lambda}_1(X), \dots, \hat{\lambda}_r(X))). \end{aligned}$$

Finally, feeding φ_g into the Chern–Weil map cw_E outputs the characteristic class

$$\varphi_g(E) = p_g(c_1(E), \dots, c_r(E)) \in H^{\mathrm{even}}(M), \tag{16}$$

where $c_k(E) = \sigma_k(\hat{\lambda}_1(F_\nabla), \dots, \hat{\lambda}_r(F_\nabla))$, and ∇ is any hermitian connection on E.[4] Note that these $c_k(E)$ differ from the Chern classes of (8), as those ones have an extra factor of $(-\frac{1}{2\pi})^k$. However

[4] While p_g is a formal power series, $\varphi_g(E)$ as defined in (16) must have finitely many nonzero summands since the cohomology ring $H^{\mathrm{even}}(M)$ is truncated. Thus $\varphi_g(E)$ genuinely belongs to $H^{\mathrm{even}}(M)$.

this normalisation is not important and from here onwards we will assume all Chern classes $c_k(E)$ are the *unnormalised* ones (i.e. without the $(-\frac{1}{2\pi})^k$ factor).

In summary, the procedure described above turns a symmetric power series g into a characteristic class $\varphi_g(E)$ for E. Moreover, by (16), $\varphi_g(E)$ is a polynomial of the Chern classes $c_k(E)$.

Example 3.3. Let $g(x_1, \dots, x_r) = \sum_{m \geq 0} \frac{s_k(x_1, \dots, x_r)}{k!}$, where $s_k = x_1^k + \dots + x_r^k$. We can write

$$g = r + \sigma_1 + \frac{1}{2}(\sigma_1^2 - 2\sigma_2) + \frac{1}{6}(\sigma_1^3 - 3\sigma_2\sigma_1 + 3\sigma_3) + \cdots ,$$

where $\sigma_k = \sigma_k(x_1, \dots, x_r)$. (See Newton's identities.) The above procedure yields a characteristic class $\varphi_g(E)$ by replacing each σ_k with $c_k(E)$. Thus

$$\varphi_g(E) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{6}(c_1(E)^3 - 3c_2(E)c_1(E) + 3c_3(E)) + \cdots .$$

This is called the *Chern character* of E and is commonly denoted $\mathrm{ch}(E)$. In fact

$$\mathrm{ch}(E) = \mathrm{tr}\exp\left(-\frac{1}{2\pi i}F_\nabla\right)$$

for any hermitian connection ∇ on E.

If f is a real-analytic function of one variable, then $g(x_1, \dots, x_r) = f(x_1)\cdots f(x_r)$ defines an element of $\mathbb{R}_{\mathrm{sym}}[[x_1, \dots, x_r]]$. The above procedure yields a characteristic class $\varphi_g(E)$, and we call f the *characteristic power series* associated to $\varphi_g(E)$.

Example 3.4. Let

$$f(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}B_{2k}}{(2k)!}x^{2k} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots ,$$

where B_k is the k-th Bernoulli number. For $g = f(x_1)\cdots f(x_r) \in \mathbb{R}_{\mathrm{sym}}[[x_1, \dots, x_r]]$, we find that

$$g = 1 + \frac{\sigma_1}{2} + \frac{\sigma_1^2 + \sigma_2}{12} + \frac{\sigma_1\sigma_2}{24} + \cdots ,$$

where g and $\sigma_1, \sigma_2, \dots$ are all evaluated at $(x_1, \dots, x_r)$. Therefore the above procedure yields the characteristic class

$$\varphi_g(E) = 1 + \frac{c_1(E)}{2} + \frac{c_1(E)^2 + c_2(E)}{12} + \frac{c_1(E)c_2(E)}{24} + \cdots \in H^{\mathrm{even}}(M).$$

This is called the *Todd class* of E and is commonly denoted $\mathrm{td}(E)$. We can write $\mathrm{td}(E) = 1 + \sum_{k \geq 1} \mathrm{td}_k(E)$, where $\mathrm{td}_k(E) \in H^{2k}(E)$. For instance,

$$\mathrm{td}_1(E) = \frac{c_1(E)}{2}, \quad \mathrm{td}_2(E) = \frac{c_1(E)^2 + c_2(E)}{12}, \quad \mathrm{td}_3(E) = \frac{c_1(E)c_2(E)}{24}.$$

The Chern character and Todd class are part of the language used to state the Hirzebruch–Riemann–Roch theorem, which generalises the Riemann–Roch theorem to complex manifolds of arbitrary dimension. This was crucial in the development of the Grothendieck–Hirzebruch–Riemann–Roch theorem and subsequently the Atiyah–Singer index theorem.

Theorem 3.5 (Hirzebruch–Riemann–Roch)**.** *Let E be a holomorphic vector bundle over a compact complex manifold X. Then the holomorphic Euler characteristic of E satisfies*

$$\chi(X, E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(TX).$$

3.2 Characteristic classes from power series: real vector bundles

Let us now assume E is a real vector bundle of rank r over M. By the remark at the end of Section 1, E is assigned characteristic classes by choosing a connection ∇ compatible with some metric on E, then defining the Chern–Weil map $\mathrm{cw}_E : I^\bullet(O(r)) \to H^{\mathrm{even}}(M)$ by

$$\mathrm{cw}_E(\varphi) = \varphi(E) := \varphi(F_\nabla, \dots, F_\nabla).$$

We want a way of coming up with characteristic classes from single-variable formal power series. This follows similarly in structure to the last subsection, although the analogue of Corollary 3.2 will look slightly more complicated. Instead of digging through all the details, we will jump straight to describing the procedure. We then follow this up with two important examples.

Let f be a real-analytic function of one variable. Then $g(x_1, \dots, x_{\lfloor r/2 \rfloor}) = f(x_1^2) \cdots f(x_{\lfloor r/2 \rfloor}^2)$ defines an element of $\mathbb{R}_{\mathrm{sym}}[[x_1, \dots, x_{\lfloor r/2 \rfloor}]]$. By the fundamental theorem of symmetric polynomials, there is a formal power series q_g so that

$$g(x_1, \dots, x_{\lfloor r/2 \rfloor}) = q_g(\sigma_1(x_1^2, \dots, x_{\lfloor r/2 \rfloor}^2), \dots, \sigma_{\lfloor r/2 \rfloor}(x_1^2, \dots, x_{\lfloor r/2 \rfloor}^2)).$$

We associate a characteristic class $\varphi_g(E) \in H^{\mathrm{even}}(M)$ to g by setting

$$\varphi_g(E) = q_g(p_1(E), \dots, p_{\lfloor r/2 \rfloor}(E)),$$

where $p_k(E) := (-1)^k c_{2k}(E_{\mathbb{C}}) \in H^{4k}(M)$. We call $p_k(E)$ the k-th *Pontryagin class* of E.

Example 3.6. Let

$$f(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 + \sum_{k \geq 1} (-1)^k \frac{2^{2k-1} - 1}{2^{2k-1}(2k)!} B_k x^k,$$

where B_k is again the k-th Bernoulli number. Writing $g(x_1, \dots, x_{\lfloor r/2 \rfloor}) = f(x_1^2) \cdots f(x_{\lfloor r/2 \rfloor}^2)$, we have

$$g = 1 - \frac{\sigma_1}{24} + \frac{-4\sigma_2 + 7\sigma_1^2}{5760} + \frac{-16\sigma_3 + 44\sigma_2\sigma_1 - 31\sigma_1^3}{967680} + \cdots,$$

where g, σ_k are evaluated at $(x_1^2, \dots, x_{\lfloor r/2 \rfloor}^2)$. Then the characteristic class arising from f according to the above construction is

$$\varphi_g(E) = 1 - \frac{p_1}{24} + \frac{-4p_2 + 7p_1^2}{5760} + \frac{-16p_3 + 44p_2p_1 - 31p_1^3}{967680} + \cdots,$$

where we have omitted E from writing. We call this the $\hat{A}$-genus of E, denoted by $\hat{A}(E)$. Writing $\hat{A}(E) = 1 + \sum_{k \geq 1} \hat{A}_k(E)$ where $\hat{A}_k(E) \in H^{4k}(M)$, we have for instance

$$\hat{A}_1(E) = -\frac{p_1}{24}, \quad \hat{A}_2(E) = \frac{-4p_2 + 7p_1^2}{5760},$$

and so on.

Example 3.7. Doing the same but starting with

$$f(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}} = \sum_{k \geq 0} \frac{2^{2k} B_{2k}}{(2k)!} x^k = 1 + \frac{x}{3} - \frac{x^2}{45} + \frac{2x^3}{945} + \cdots, \tag{17}$$

the characteristic class we obtain is called the L-genus of E, denoted $L(E)$. Likewise decomposing $L(E) = 1 + \sum_{k \geq 1} L_k(E)$ where $L_k(E) \in H^{4k}(M)$, the first few terms turn out to be

$$L_1(E) = \frac{p_1}{3}, \quad L_2(E) = \frac{7p_2 - p_1^2}{45}, \quad L_3(E) = \frac{62p_3 - 13p_1p_2 + 2p_1^3}{945}, \tag{18}$$

where we have again omitted E from writing.

3.3 The Hirzebruch signature theorem

There is a wealth of theory on the genera introduced in the previous examples, and it is far beyond our scope to give a meaningful discussion of the deep results in this area. However, we are able to sketch a proof of one such result, the *Hirzebruch signature theorem.* To state it, we must define the *signature* of a manifold. Recall that if V is a finite-dimensional real vector space and $q : V \times V \to \mathbb{R}$ is a symmetric bilinear form, then the *signature* of q is the number of positive eigenvalues of q less the number of negative eigenvalues when considered as a matrix in some basis. This quantity is basis-independent by Sylvester's law of inertia.

Definition 3.8. The *signature* of a compact oriented $4n$-dimensional manifold M, denoted $\sigma(M)$, is the signature of the symmetric bilinear form

$$H^{2n}(M, \mathbb{R}) \times H^{2n}(M, \mathbb{R}) \to \mathbb{R}, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

If $\dim M$ is not divisible by 4, then we assign $\sigma(M) = 0$.

To state the Hirzebruch signature theorem, we also need to define the quantity $\mathbf{L}(M)$ for a $4n$-dimensional manifold M by

$$\mathbf{L}(M) = \int_M L_n(TM),$$

where $L_n(TM) \in H^{4n}(M)$ was introduced in Example 3.7. If $4 \nmid \dim M$, we assign $\mathbf{L}(M) = 0$.

Theorem 3.9 (Hirzebruch signature theorem)**.** *For any compact oriented manifold M we have*

$$\sigma(M) = \mathbf{L}(M). \tag{19}$$

Originally proved in 1953 [Hir53], this theorem was used in the (original) proof of Hirzebruch–Riemann–Roch, i.e. Theorem 3.5. It also enabled Milnor to construct the first exotic 7-spheres [Mil56]. Yet, it is a surprising result in its own right: it shows that $\mathbf{L}(M)$ is always an integer (as $\sigma(M)$ is clearly an integer). A priori, $\mathbf{L}(M)$ is only known to be a real number.

Our proof sketch of Theorem 3.9 will employ the following lemma, which is immediate from replicating the proof of Lemma 2.6. Here $p(E)$ is the *total Pontryagin class* of a real vector bundle E, defined by $p(E) = 1 + p_1(E) + \ldots + p_r(E)$ if E has rank r. Like the total Chern class, this encodes all information about the individual Pontryagin classes.

Lemma 3.10. *If E and E' are real vector bundles, then*

$$p(E \oplus E') = p(E)p(E'), \quad L(E \oplus E') = L(E)L(E').$$

Proof sketch for Theorem 3.9. We follow [MS74, Theorem 19.4]. There are two parts.

Part 1 (sketch): reducing the problem. The proof uses oriented cobordism. Two oriented manifolds N and N' of equal dimension are *oriented cobordant* if there exists an oriented manifold of one dimension higher whose (oriented) boundary is $N + (-N')$, where $+$ denotes disjoint union and $-N'$ is N' with the opposite orientation. Oriented cobordism defines an equivalence relation on the set of oriented manifolds of any given dimension, and we denote by Ω_m^{SO} the set of oriented cobordism classes of manifolds of dimension m.

Write $\Omega_*^{\mathrm{SO}} = (\Omega_0^{\mathrm{SO}}, \Omega_1^{\mathrm{SO}}, \ldots)$. One checks that $(\Omega_*^{\mathrm{SO}}, +, \times)$ is a graded commutative ring under the operations of disjoint union and Cartesian product. Within each Ω_m^{SO}, the additive identity is the equivalence class of m-manifolds that are oriented boundaries of $(m+1)$-manifolds. By a result of Thom (see [MS74, Lemma 19.3]), the signature operator σ satisfies

$$\sigma(M + M') = \sigma(M) + \sigma(M'), \quad \sigma(M \times M') = \sigma(M)\sigma(M'),$$

and $\sigma(N) = 0$ if N is an additive identity in Ω_*^{SO}. Therefore, σ descends to a ring homomorphism $\Omega_*^{\mathrm{SO}} \to \mathbb{Z}$. As Ω_*^{SO} can be seen as a graded $\mathbb{Z}$-algebra, tensoring with $\mathbb{Q}$ turns σ into an algebra homomorphism $\sigma : \Omega_*^{\mathrm{SO}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$.

Let M be as in the theorem and $J = (j_1, \ldots, j_k)$ a multi-index with $\sum j_i = n$. Define the *Pontryagin number* $p_J(M)$ by

$$p_J(M) = \int_M p_{j_1}(TM) \wedge \ldots \wedge p_{j_k}(TM).$$

One can show that p_J is an oriented cobordism invariant for each J. Therefore, p_J descends to a map $\Omega_*^{\mathrm{SO}} \to \mathbb{R}$, and by extension $\Omega_*^{\mathrm{SO}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$. But $\mathbf{L}(M)$ is a polynomial in the $p_J(M)$. Therefore $\mathbf{L}$ also descends to a map $\Omega_*^{\mathrm{SO}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{R}$.

From the last two paragraphs, σ and $\mathbf{L}$ both filter through oriented cobordism, hence descend to maps on $\Omega_*^{\mathrm{SO}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Thom also showed that (see [MS74, Theorem 18.9])

$$\Omega_*^{\mathrm{SO}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]$$

where the right-hand side is a polynomial ring. Therefore, to demonstrate the equality (19) for any compact oriented manifold M, it suffices to check it for $M = \mathbb{CP}^{2n}$, $n \in \mathbb{N}$. Since Lemma 2.9 shows that $\sigma(\mathbb{CP}^{2n}) = 1$, we need only check that $\mathbf{L}(\mathbb{CP}^{2n}) = 1$ too.

Part 2 (less sketchy): checking $\mathbf{L}(\mathbb{CP}^{2n}) = 1$. We begin this step with a general fact. If E is a complex rank r vector bundle $E_{\mathbb{R}}$ is its realification, then

$$1 - p_1 + p_2 - \cdots \pm p_r = (1 - c_1 + c_2 - \cdots \pm c_r)(1 + c_1 + \cdots + c_r),$$

where the p_k are the Pontryagin classes $p_k(E_{\mathbb{R}})$, and the c_ℓ are the Chern classes $c_\ell(E)$. This can be verified using the definition of Pontryagin classes (see the line before Example 3.6), and Lemmas 2.6 and 2.7. Using this with $E = T\mathbb{CP}^{2n}$ and applying Theorem 2.3, we get

$$1 - p_1 + p_2 - \cdots \pm p_{2n} = (1 + \alpha)^{2n+1}(1 - \alpha)^{2n+1} = (1 - \alpha^2)^{2n+1},$$

where $\alpha = c_1(\gamma^1)$ and p_k means $p_k((T\mathbb{CP}^{2n})_{\mathbb{R}})$. This implies the total Pontryagin class of $(T\mathbb{CP}^{2n})_{\mathbb{R}}$ is given by

$$p((T\mathbb{CP}^{2n})_{\mathbb{R}}) = (1 + \alpha^2)^{2n+1}. \tag{20}$$

We will use this to compute $L(T\mathbb{CP}^{2n})$ and thereby $\mathbf{L}(\mathbb{CP}^{2n})$. If E is a real vector bundle over $\mathbb{CP}^{2n}$ with $p(E) = 1 + \alpha^2$, then by the vanishing of $p_2(E), p_3(E), \ldots$, we get

$$L(E) = \frac{\alpha}{\tanh \alpha}.$$

(Compare (17) and (18) to convince oneself of this.) By Lemma 3.10, $E^{\oplus(2n+1)}$ has

$$p(E^{\oplus(2n+1)}) = (1 + \alpha^2)^{2n+1}, \quad L(E^{\oplus(2n+1)}) = \left(\frac{\alpha}{\tanh \alpha}\right)^{2n+1}.$$

Since the L-genus depends only on the Pontryagin classes, and $(T\mathbb{CP}^{2n})_{\mathbb{R}}$ and $E^{\oplus(2n+1)}$ have the same Pontryagin classes by (20) and the above equation, they must also have the same L-genera. Thus

$$L((T\mathbb{CP}^{2n})_{\mathbb{R}}) = \left(\frac{\alpha}{\tanh \alpha}\right)^{2n+1}. \tag{21}$$

Since $L_n((T\mathbb{CP}^{2n})_{\mathbb{R}})$ is the component of this belonging to $H^{4n}(\mathbb{CP}^{2n})$, it is the α^{2n} term in the power series expansion of (21). Now Lemma 2.9 says $\int_{\mathbb{CP}^{2n}} \alpha^{2n} = 1$, so $\mathbf{L}(\mathbb{CP}^{2n})$ is simply the coefficient of α^{2n}. We will compute this by substituting α for a complex variable z and applying Cauchy's integral formula, so that

$$\mathbf{L}(\mathbb{CP}^{2n}) = \frac{1}{2\pi i}\oint_C \left(\frac{z}{\tanh z}\right)^{2n+1} \frac{dz}{z^{2n+1}} = \frac{1}{2\pi i}\oint_C \frac{dz}{(\tanh z)^{2n+1}},$$

where C is a small contour about the origin in $\mathbb{C}$. Substituting $u = \tanh z$ and $dz = du/(1-u^2) = (1 + u^2 + u^4 + \ldots)du$, this gives

$$\mathbf{L}(\mathbb{CP}^{2n}) = \frac{1}{2\pi i}\oint_{\tanh C} \frac{1 + u^2 + u^4 + \cdots}{u^{2n+1}}\, du = 1,$$

where the second equality is by the residue theorem. This is what we needed to show. □

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